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Koen Thas

Symmetry in Finite Generalized Quadrangles

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To Nathalie, my Parents and my Sister

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Introduction

Lenz-Barlotti-Type Classifications and Symmetry

A (finite) *generalized n-gon* $\mathcal{S}$ *of order* (s,t), $n \geq 2$, $s > 0$, $t > 0$ (where all the parameters are finite), is a 1-$(v, s+1, t+1)$ design whose incidence graph has girth $2n$ and diameter n. A *generalized polygon* is a generalized n-gon for some n. Generalized polygons were introduced by J. Tits [160] in his celebrated triality paper "Sur la trialité et certains groupes qui s'en déduisent" (*Inst. Hautes Etudes Sci. Publ. Math.* **2** (1959), 13–60 [160])[1] in order to understand better the Chevalley groups of rank 2. The classical examples arise from groups with a (split) BN-pair of rank 2, for which the Weyl group is a dihedral group. If $n = 3$, then $\mathcal{S}$ is also called a *projective plane*, and in that case $s = t$ and the plane is said to be *of order* s. If $n = 4$, then one speaks of a *generalized quadrangle*.

In *Finite Geometries* [27], P. Dembowski wrote that an alternative approach to the study of projective planes began with the paper *Homogeneity of projective planes, R. Baer* (1942) [6], in which the close relationship between Desargues' theorem and the existence of central collineations was pointed out. Baer's notion of (p, L)-*transitivity*, corresponding to this relationship, proved to be extremely fruitful; it provided a better understanding of coordinate structures and it led eventually to the only coordinate-free (and hence geometrically satisfactory) classification of projective planes existing today, namely the classification by H. Lenz in *Kleiner Desarguesscher Satz und Dualität in projektiven Ebenen* (1954) [64] and A. Barlotti in *Sulle possibili configurazioni del sistema delle coppie punto-retta* (A, a) *per cui un piano grafico risulta* (A, a)*-transitivo* (1958) [9], see also [111]. Due to deep discoveries in finite group theory, the analysis of this classification has been particularly penetrating for finite projective planes in recent years.

For generalized quadrangles (GQ's), J. A. Thas and H. Van Maldeghem [137] gave a (first) definition of Desargues configurations and proved a result analogous to the theorem of R. Baer for projective planes (i.e., it asserts that a local configurational condition holds if and only if a certain locally defined collineation group is as large as possible). In the theory of projective planes, the configuration involves

[1] For more on generalized n-gons, see [46, 91, 164].

two triangles in perspective from a point (or a line); in the theory of generalized quadrangles the configuration involves two quadrilaterals in perspective from a 'root' (or 'panel', see below). In H. Van Maldeghem, J. A. Thas and S. E. Payne [165] there was a second approach to the problem, as follows. Let (p, L) be a flag of a generalized quadrangle $\mathcal{S}$ (i.e. an incident point-line pair). A collineation θ of $\mathcal{S}$ is called a (p, L)-*collineation* if θ fixes each point on L and each line through p. The group $G(p, L)$ of all (p, L)-collineations acts semiregularly on the set of points of $\mathcal{S}$ collinear with u but not on L, where $u \neq p$ is a point on L, and dually on the set of lines of $\mathcal{S}$ concurrent with N but not passing through p, where $N \neq L$ is a line through p. If $G(p, L)$ acts regularly on these sets, then $\mathcal{S}$ is said to be (p, L)-*transitive*. It is easy to see that each *Moufang generalized quadrangle* (see further) is (p, L)-transitive for all flags (p, L). The authors of [165] proved that for finite $\mathcal{S}$ the converse also holds; a finite generalized quadrangle $\mathcal{S}$ is (p, L)-transitive for all its flags (p, L) if and only if $\mathcal{S}$ is Moufang and hence classical or dual classical by the following celebrated and deep group theoretical result of P. Fong and G. M. Seitz [32, 33], essentially on finite groups with a split BN-pair of rank 2:

Theorem A (P. Fong and G. M. Seitz [32, 33]). *A thick finite Moufang GQ is classical or dual classical.* ■

(We also refer the reader to the recent work of J. Tits and R. Weiss [163], which also treats the infinite case.)

In Chapter 9 of [91], an almost complete elementary proof of the previous result of P. Fong and G. M. Seitz was given for the finite case; however, one case (*) remained open. First recall that an *axis of symmetry* L of a GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$, is a line for which there is a group of automorphisms of $\mathcal{S}$ fixing each line meeting L, that acts regularly on the set of points not on L which are on an arbitrary line ($\neq L$) which intersects L.

PROBLEM. *Classify all finite GQ's of order (s, s^2), $s > 1$, each line of which is an axis of symmetry.* (*)

In the appendix of the extensive survey [143] (see Section 7.14 of the present work), we solved a much more general version of that case with the use of the classification of finite split BN-pairs of rank 1 and results of [147]. The proof is/was still not elementary in the sense of the program of S. E. Payne and J. A. Thas [91], but much more elementary than [32, 33]. In [154], we generalized the result of [143], solving thusly a conjecture of W. M. Kantor, the proof of which will be given in this monograph, see Section 12.2:

Theorem B (See §12.2). *A finite generalized quadrangle of order (s, t), $s \neq 1 \neq t$, is isomorphic to $\mathcal{Q}(5, s)$ if and only if there are three distinct axes of symmetry U, V and W for which U does not meet any line of $\{V, W\}^{\perp\perp}$.* ■

W. M. Kantor provided a solution of (*) in [54], depending on 4B,C of P. Fong and G. M. Seitz [32], and also on the classification of finite groups admitting a split BN-pair of rank 1. Very recently, J. A. Thas [131] gave a complete solution of (*) without the use of group theory, as a corollary of a more general result; he has shown that a finite *translation generalized quadrangle* (introduced by J. A. Thas [115] in 1974, cf. Chapter 2), each line of which is regular, is classical, except, possibly, when the kernel of the translation generalized quadrangle is the finite field with two elements $\mathbf{GF}(2)$. If the kernel is $\mathbf{GF}(2)$, then the generalized quadrangle is classical if there is another translation point (see Chapter 8).

As a geometric counterpart to (p, L)-transitivity, S. E. Payne, J. A. Thas and H. Van Maldeghem introduced the notion of a (p, L)*-Desarguesian generalized quadrangle*, and proved that a finite generalized quadrangle is (p, L)-Desarguesian if and only if it is (p, L)-transitive.
Before proceeding, let us first recall that a *whorl* about a point p in a GQ $\mathcal{S}$ is a collineation θ of $\mathcal{S}$ which fixes p linewise. If no point non-collinear with p is fixed by the whorl, or if $\theta = \mathbf{1}$, then θ is an *elation* about p. *Whorls* and *elations* about a line are defined dually.
A *panel* (or *root*) of a generalized quadrangle $\mathcal{S} = (P, B, I)$ is an element (p, L, q) of $P \times B \times P$ for which $pILIq$ and $p \neq q$. Dually, one defines *dual panels* (or *dual roots*). If (p, L, q) is a panel of the GQ $\mathcal{S}$, then a (p, L, q)*-collineation* of $\mathcal{S}$ is a whorl about p, L and q. Such a (p, L, q)-collineation is also called a *root-elation*. A panel (p, L, q) of a GQ of order (s, t), $s \neq 1 \neq t$, is called *Moufang*, or the GQ is called (p, L, q)*-transitive*, if there is a group of (p, L, q)-collineations of size s. A line L is *Moufang* if every panel of the form (p, L, q) is Moufang. A GQ is *half Moufang* if every panel *or* every dual panel is Moufang, and a GQ is a *Moufang GQ* if every panel *and* every dual panel is Moufang.

Theorem C (J. A. Thas, S. E. Payne and H. Van Maldeghem [139]). *Any thick finite half Moufang GQ is Moufang.* ■

This result was recently obtained also for the infinite case by K. Tent in [113]. By Theorem C every half Moufang GQ is automatically Moufang, and hence classical or dual classical by Theorem A.
Now let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s \neq 1 \neq t$, and let H be an automorphism group of $\mathcal{S}$. For points p, q of $\mathcal{S}$, denote by $H_{p,q}$ the stabilizer of both p and q in H. Similarly for lines. Then $\mathcal{S}$ is called *half pseudo Moufang* (with respect to H) if *either* Property (PM) or Property (PM′) (below) is satisfied.

(PM) For every panel (p, L, q), with $p, q \in P$ and $L \in B$, there is a normal subgroup $H(p, q)$ of $H_{p,q}$ of elations about both p and q which acts regularly on the set of points that are incident with any line MIp, respectively $M'Iq$, $M \neq L \neq M'$, and different from p, respectively q. The group $H(p, q)$ will be referred to as a *pseudo elation group*.

(PM′) For every dual panel (L, p, M), with $L, M \in B$ and $p \in P$, there is a normal subgroup $H(L, M)$ of $H_{L,M}$ of elations about both L and M which acts regularly on the set of lines that are incident with an arbitrary point xIL, respectively yIM, $x \neq p \neq y$, and different from L, respectively M. The group $H(L, M)$ will also be referred to as a *pseudo elation group.*

A GQ is called *pseudo Moufang* (with respect to H) if *both* Properties (PM) *and* (PM′) hold. We will sometimes write 'HPMGQ' instead of 'half pseudo Moufang generalized quadrangle' for the sake of convenience, and we always assume that the corresponding group is H. Recently, the author and H. Van Maldeghem [159] have obtained the result that every finite thick HPMGQ is a classical or dual classical generalized quadrangle. Conversely, every (finite) classical or dual classical generalized quadrangle is an HPMGQ with respect to any group H containing the little projective group (that is, the group generated by all root-elations), and the pseudo elation groups are independent of H. In particular, every pseudo elation group is a group of root-elations.
A GQ is called a *strong elation generalized quadrangle (SEGQ)* if for each of its points there is a group of automorphisms fixing the point linewise and acting regularly on the set of points which are not collinear with that point. It was a well known open problem to classify the finite SEGQ's without the classification of finite simple groups. As each half Moufang generalized quadrangle is an SEGQ (up to point-line duality), this problem also generalizes the problem considered in [139]. In [159], the author and H. Van Maldeghem have obtained that each finite SEGQ is classical or dual classical without the classification of finite simple groups, as a corollary of the more general result that a GQ which admits for each of its points a group of automorphisms fixing it linewise and acting transitively on the points not collinear with that point — a GQ with this property or its dual property is called 'half 2-Moufang' — is classical or dual classical (see also the (partial) survey [158]):

Theorem D (K. Thas and H. Van Maldeghem [159]). *A thick finite half* 2-*Moufang generalized quadrangle is classical or dual classical, and conversely.* ■

The natural analogue of (p, L)-transitivity, $p \not\mathrm{I} L$, for projective planes in the theory of generalized quadrangles is that of '(x, y)-transitivity':

A GQ is (x, y)-transitive, *where x and y are non-collinear points, if there is a group H of whorls about x and y (such automorphisms are usually called* generalized homologies with centers x and y*) which acts transitively on the points incident with at least one line M through x, not contained in $\{x\} \cup \{x, y\}^{\perp}$.*

The main configurational result in this Barlotti-type theory for GQ's is the following:

Theorem E (J. A. Thas [118, 119]). *Each thick finite GQ $\mathcal{S}$ which is (x, y)-transitive for each $x \not\sim y$ in $\mathcal{S}$ (where x and y are points), is classical.* ■

A GQ is called *quasi-transitive* (w.r.t. points) if for each pair of non-collinear points (x, y), the group of generalized homologies with centers x and y acts transitively on the set of lines through any (fixed) point of $\{x, y\}^\perp$ not containing a point of $\{x, y\}$. By a recent result of the author and H. Van Maldeghem, the finite quasi-transitive GQ's are known:

Theorem F (K. Thas and H. Van Maldeghem [158]). *Each thick finite quasi-transitive GQ (w.r.t. points) of order (s, t), is isomorphic to one of $\mathcal{Q}(4, s)$, $H(3, s)$, and conversely.* ■

In this book, only a Lenz-type classification will be considered. As the Barlotti part of the classification (clearly) needs a completely different approach, we see both parts as being different theories rather than a unified one (such as in the case of projective planes). A Barlotti-type classification is being prepared by the author, cf. [157]. We still prefer the term "Lenz-Barlotti classification" instead of "Lenz classification", though.

Despite the promising results mentioned, a 'good' Lenz classification based on subconfigurations of flags (p, L), respectively panels (p, L, q), for which the generalized quadrangle is (p, L)-transitive, respectively (p, L, q)-transitive, seems (quite) far away and would yield many open classes very hard to deal with. We present the following alternative.

The relation between the Moufang condition and the notion of axis of symmetry is given by the following observation (cf. Chapter 2):

A line L of a generalized quadrangle $\mathcal{S}$ is an axis of symmetry if it is regular and if there is a pair of distinct points (p, q) both incident with L for which the generalized quadrangle is (p, L, q)-transitive.

In our Master Thesis [141], we started to investigate the generalized quadrangles with concurrent (i.e. intersecting) axes of symmetry, as generalizations of translation generalized quadrangles (which are natural generalizations of translation planes [46]), and there we obtained a first version of a Lenz-Barlotti classification for finite generalized quadrangles, cf. Theorem G below. Part of that work resulted in the papers [142], [150] — where we introduced *semi quadrangles* to have a better understanding of the GQ's with concurrent axes of symmetry (cf. Appendix A) — and the extensive paper [148] (cf. Chapter 6).
Generalized quadrangles with non-concurrent axes of symmetry are also called *span-symmetric generalized quadrangles (SPGQ's)*, and in the series of papers [145], [147], [153], many problems in the theory of SPGQ's were completely solved, including a complete classification of SPGQ's of order s, which was independently obtained in [56] by W. M. Kantor, and thus solving an open problem from 1980. A complete theory for SPGQ's, including the aforementioned and many new results, is contained in the Chapters 7, 8, 10, 11 and 12 of the present work.

Here, we classify generalized quadrangles based on the possible subconfigurations of axes of symmetry.

We will develop that classification in such a way that

(a) it provides a general(ized) theory for those generalized quadrangles having a point x each line through which is an axis of symmetry (that is, a translation point); such a GQ is called a 'translation generalized quadrangle' with 'translation point' or 'base-point' x;

(b) the theory of the generalized quadrangles which have non-concurrent axes of symmetry (the SPGQ's), is worked out in depth;

(c) (a) and (b) are unified in the theory of *(semifield) flocks* (cf. Section 2.7) of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3, q)$;

(d) (a), (b) and (c) provide a 'blueprint' for the classification of all translation generalized quadrangles.

Final Note. Each known GQ or its point-line dual admits axes of symmetry or is constructed from a GQ with axes of symmetry, the only known exceptions being the classical example $H(4, q^2)$ and its point-line dual $H(4, q^2)^D$. The GQ $H(4, q^2)$ has the property though that for each point x, there is a group of $q+1$ symmetries with center x (and so the dual property holds for $H(4, q^2)^D$). It is hence fair to say that axes of symmetry, respectively symmetries, play a central role in finite GQ theory.

Organization of the Book

This book is organized as follows. In Chapter 1, the necessary basics are introduced. A theory for elation and translation generalized quadrangles is described in Chapter 2, and new results are introduced in order to obtain a clear view on symmetry. Chapter 3 is a detailed synopsis of the known (constructions of) finite generalized quadrangles. We review the connection between nets and generalized quadrangles with a regular point in Chapter 4, and work out the relation(s) between subnets and subquadrangles (and applications on collineations). In Chapter 5, the generalized quadrangles which have no axes of symmetry are investigated, and we do a thorough study of generalized quadrangles which have some axes of symmetry incident with a fixed point in Chapter 6. In Chapter 7, we introduce a general theory for generalized quadrangles which have non-concurrent axes of symmetry. We also provide a proof of Part (*) in the Moufang Theorem without relying on the classification of finite groups with a split BN-pair of rank 2. Then, in Chapter 8, the generalized quadrangles with distinct translation points are classified.

The configurational structure theorem that will be obtained in Chapter 9, and which was essentially obtained in our Master Thesis [141], is:

Theorem G (See K. Thas [141] and Chapter 9). *Suppose $\mathcal{S} = (P, B, I)$ is a generalized quadrangle of order (s, t), $s \neq 1 \neq t$. Then we have one of the following possibilities.*

I. *$\mathcal{S}$ contains no axis of symmetry.*

II. *Every axis of symmetry is incident with some fixed point $p \in P$.*

III. *There is a line $L \in B$ which is not an axis of symmetry such that every point qIL is incident with exactly $k+1$ axes of symmetry, k a constant in $\{0, s-1, s^2-1\}$, and there are no other axes of symmetry.*

IV. *There is an axis of symmetry $L \in B$ such that every point qIL is incident with exactly $k+1$ axes of symmetry, k a constant in $\{1, s, s^2\}$, and there are no other axes of symmetry.*

V. *Suppose none of the previous cases holds. Define an incidence structure $\mathcal{S}' = (P', B', I')$ as follows.*

- LINES *are of two types:*
  (1) *the axes of symmetry of $\mathcal{S}$;*
  (2) *the lines of $\mathcal{S}$ not of Type (1) such that each point of such a line is incident with an axis of symmetry.*
- *The* POINTS *of $\mathcal{S}'$ are the points lying on the axes of symmetry.*
- INCIDENCE *is the restriction of I to $(P' \times B') \cup (B' \times P')$.*

Then $\mathcal{S}'$ is a subGQ of $\mathcal{S}$, every point of $\mathcal{S}'$ is incident with a constant number $k+1$ of axes of symmetry, and one of the following possibilities holds.

(i) *$k = 0$ and $\mathcal{S}$ has a regular spread $\mathbf{T}_N$ of which any line is an axis of symmetry of $\mathcal{S}$. Furthermore, the group $Aut(\mathcal{S})$ acts transitively on the lines of $\mathbf{T}_N$, $\mathcal{S}' = \mathcal{S}$ and $t = s^2$.*

(ii) *We have that $k = 1$, and there are two possibilities.*
  (a) *$\mathcal{S} = \mathcal{S}'$ and $\mathcal{S}$ is of order (s, s^2).*
  (b) *$\mathcal{S}'$ is a grid with parameters $s+1, s+1$, and hence each line of $\mathcal{S}'$ is an axis of symmetry of $\mathcal{S}$. Moreover, $\mathcal{S}$ is a GQ of order (s, s^2).*

(iii) *$2 \leq k < t$ and one of the following possibilities holds.*
  (a) *$\mathcal{S}' = \mathcal{S}$ and $\mathcal{S}$ is of order (s, s^2).*
  (b) *$k = s$, $\mathcal{S}' \neq \mathcal{S}$, $\mathcal{S}' \cong \mathcal{Q}(4, s)$ and $t = s^2$.*

VI. *Every line of $\mathcal{S}$ is an axis of symmetry and then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4, s)$ or $\mathcal{Q}(5, s)$.*

Theorem G will indicate precisely how we should define certain "symmetry-classes", a term which will be therefore used prior to Chapter 9 without causing confusion.

Then, in Chapter 10, we prove that a translation generalized quadrangle which is the translation dual of the point-line dual of a flock GQ in odd characteristic, always has a line of translation points — in this way completing the classification of Chapter 8. In Chapter 11, two strong and very general classification and characterization results are obtained. In Chapter 12, we study the GQ's which are described in **V** of Theorem G. It contains generalizations of known theorems and a complete solution of a conjecture of W. M. Kantor.
In Chapter 13, we give a table which overviews a finalized version of the classification, presenting 36 distinct symmetry-classes. Based on observations made in the body of the present work, we construct a conjectural 'blueprint' for the classification of all finite translation generalized quadrangles, in Chapter 14. We introduce a new abstract point-line geometry called 'semi quadrangle' in Appendix A — motivated by certain observations of Chapter 6 — and study these structures in great detail.

We have included two indexes in the text, namely a general (subject) index and an explanatory index of notations, in which the notations and mathematical abbreviations which frequently and consistently occur in this work are defined.

Note on the (subject) index. If a notion is defined in the introduction of this work, but also in the body of the text, then usually we only refer to the definition appearing in the text (exceptions are made sometimes for reasons of convenience). Furthermore, some miscellaneous (often undefined) notions and notations are inserted in the subject index for the comfort of the reader.

The book contains tables on the sizes of some groups and some of their Schur multipliers, see Appendix B, and an extensive bibliography.

I have also included several conjectures on open problems of varying (sometimes very high) difficulty.

Although the main contribution of this work is the classification and its proof, we have organized it in such a way that it is suited for various other purposes. Often, the chapters are written in a more general setting then needed for the classification; as such, they can be viewed and used as being independent of it.

Proofs will be given only for theorems which have direct implications for *the possible configurations of axes of symmetry* in generalized quadrangles.

If a large part of (proofs in) a chapter was already published elsewhere, comments on that matter will be made in the beginning of that chapter.
If only few (proofs of) results in a chapter were previously published (if at all), the appropriate references will be made when stating the results.

The reader should be familiar with the basics of finite projective (Galois) geometry (see, e.g., [43, 45]), and have standard knowledge of (permutation) group theory

(see, for example, [3, 35]). Also, it could be helpful to have a standard reference on (finite) incidence geometry (such as [27]), and in particular on generalized quadrangles ([91]), at hand.

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Chapter 1

Finite Generalized Quadrangles

In this chapter, it is our purpose to introduce the reader to the (first) basics of finite generalized quadrangles that will be needed for the present work. Most of the results (which are all given without proofs) are taken from the monograph *Finite Generalized Quadrangles* by S. E. Payne and J. A. Thas [91] — which will be denoted by "FGQ" throughout, and we refer to *loc. cit.* for primary references. The chapter can be viewed as a combinatorial introduction to finite generalized quadrangles.

1.1 Standard Theory

A (finite) *generalized quadrangle (GQ) of order* (s, t) (or *with parameters* (s, t)), $s \geq 1$ and $t \geq 1$ and $s, t \in \mathbb{N}$, is a point-line incidence structure $\mathcal{S} = (P, B, I)$ in which P and B are disjoint (non-empty) sets of objects called 'points' and 'lines' respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms.

(1) Each point is incident with $t + 1$ lines, and two distinct points are incident with at most one line.

(2) Each line is incident with $s + 1$ points, and two distinct lines are incident with at most one point.

(3) If p is a point and L is a line not incident with p, then there is a unique point-line pair (q, M) such that $pIMIqIL$.

If $s = t$, then $\mathcal{S}$ is also said to be 'of order s'.

There is a point-line duality for GQ's of order (s, t) for which in any definition or theorem the words 'point' and 'line' are interchanged, and also the parameters. Normally, we assume without further notice that the dual of a given theorem or definition has also been given.

Often we will identify a line with the set of points incident with it. Also, sometimes a line set will be identified with the set of points which are incident with the lines of that set (rather than seeing it as a set of point sets).

A *grid*, respectively *dual grid*, is an incidence structure $\Gamma = (P, B, I)$ with $P = \{x_{ij} \parallel i = 0, 1, \ldots, s_1 \text{ and } j = 0, 1, \ldots, s_2\}$, $s_1, s_2 > 0$, respectively $B = \{L_{ij} \parallel i = 0, 1, \ldots, t_1 \text{ and } j = 0, 1, \ldots, t_2\}$, $t_1, t_2 > 0$, with $B = \{L_0, L_1, \ldots, L_{s_1}, M_0, M_1, \ldots, M_{s_2}\}$, respectively $P = \{x_0, x_1, \ldots, x_{t_1}, y_0, y_1, \ldots, y_{t_2}\}$, $x_{ij} I L_k$ if and only if $i = k$, respectively $L_{ij} I x_k$ if and only if $i = k$, and $x_{ij} I M_k$ if and only if $j = k$, respectively $L_{ij} I y_k$ if and only if $j = k$. If Γ is a grid, respectively dual grid, and if s_1 and s_2, respectively t_1 and t_2, are as above, then we say that Γ has *parameters* $s_1 + 1, s_2 + 1$, respectively *parameters* $t_1 + 1, t_2 + 1$.[1] A grid, respectively dual grid, with parameters $s_1 + 1, s_2 + 1$, respectively with parameters $t_1 + 1, t_2 + 1$, is a GQ if and only if $s_1 = s_2$, respectively $t_1 = t_2$. It is clear that the grids, respectively dual grids, with $s_1 = s_2$, respectively $t_1 = t_2$, are the GQ's with $t = 1$, respectively $s = 1$. Sometimes we will speak of an '$(s_1 + 1) \times (s_2 + 1)$-grid', respectively 'dual $(t_1+1) \times (t_2+1)$-grid', instead of a 'grid with parameters s_1+1, s_2+1', respectively 'dual grid with parameters $t_1 + 1, t_2 + 1$'. A GQ is called *thick* if every point is incident with more than two lines and if every line is incident with more than two points. Otherwise, a GQ is called *thin*. So a thin GQ of order $(s, 1)$ is just a grid with parameters $s + 1, s + 1$.
We call a grid with parameters $k, s + 1$ of a thick GQ $\mathcal{S}$ of order (s, t), which is not contained in any grid of $\mathcal{S}$ with parameters $k', s + 1$ with $k' > k$, a *complete* $k \times (s + 1)$*-grid*.

Let $\mathcal{S} = (P, B, I)$ be a (finite) generalized quadrangle of order (s, t), $s \neq 1 \neq t$. Then $|P| = (s + 1)(st + 1)$, $|B| = (t + 1)(st + 1)$ and $s + t$ divides $st(s + 1)(t + 1)$ (see 1.2.1 and 1.2.2 of FGQ). Also, $s \leq t^2$ [41, 42] and, dually, $t \leq s^2$, and we will refer to both inequalities as being the "Inequality of Higman".

Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$ and say that p and q are *collinear*, provided that there is some line L so that $pILIq$ (so $p \not\sim q$ means that p and q are *not* collinear). Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or *non-concurrent*. If $p \neq q \sim p$, the line incident with both is denoted by pq, and if $L \sim M \neq L$, the point which is incident with both is sometimes denoted by $L \cap M$.
For $p \in P$, put $p^\perp = \{q \in P \parallel q \sim p\}$, and note that $p \in p^\perp$. For a pair of distinct points $\{p, q\}$, the *trace* of $\{p, q\}$ is defined as $p^\perp \cap q^\perp$, and we denote this set by $\{p, q\}^\perp$. Then $|\{p, q\}^\perp| = s + 1$ or $t + 1$, according as $p \sim q$ or $p \not\sim q$. We use the same notations for lines. More generally, if $A \subseteq P$ or $A \subseteq B$, $A^\perp$ is defined by

$$A^\perp = \bigcap \{X^\perp \parallel X \in A\}.$$

[1] In FGQ, a slightly different definition for the parameters of a grid or dual grid is used. This will not lead to confusion, however.

For $p \neq q$, the *span* of the pair $\{p, q\}$ is $sp(p, q) = \{p, q\}^{\perp\perp} = \{r \in P \parallel r \in s^{\perp}$ for all $s \in \{p, q\}^{\perp}\}$. When $p \not\sim q$, then $\{p, q\}^{\perp\perp}$ is also called the *hyperbolic line* defined by p and q, and $|\{p, q\}^{\perp\perp}| = s + 1$ or $|\{p, q\}^{\perp\perp}| \leq t + 1$ according as $p \sim q$ or $p \not\sim q$. More generally, for $A \subseteq P$ or $A \subseteq B$, $A^{\perp\perp}$ is defined by

$$A^{\perp\perp} = (A^{\perp})^{\perp}.$$

If $p \sim q$, $p \neq q$, or if $p \not\sim q$ and $|\{p, q\}^{\perp\perp}| = t + 1$, then we say that the pair $\{p, q\}$ is *regular*. The point p is *regular* provided $\{p, q\}$ is regular for every $q \in P \setminus \{p\}$. Regularity for lines is defined dually. One easily proves that either $s = 1$ or $t \leq s$ if $\mathcal{S}$ has a regular pair of non-collinear points. A point p is *coregular* provided each line incident with p is regular. Dually, one defines *coregular lines.*
A *flag* of a GQ is an incident point-line pair. If (p, L) is a non-incident point-line pair of a GQ (i.e. an *anti-flag*), then by $[p, L]$ we denote the unique line of the GQ which is incident with p and concurrent with L. Sometimes, we will also use the notation $proj_p L$, and, dually, $proj_L p$.
A *panel* or *root* of a generalized quadrangle $\mathcal{S} = (P, B, I)$ is an element (p, L, q) of $P \times B \times P$ for which $pILIq$ and $p \neq q$. Dually, one defines *dual panels* and *dual roots*.

Finally, if $\mathcal{S}$ is a GQ, then by $\mathcal{S}^D$ we denote its point-line dual.

1.2 Automorphisms of Finite Generalized Quadrangles

1.2.1 Automorphisms

A *collineation* or *automorphism* of a generalized quadrangle $\mathcal{S} = (P, B, I)$ is a permutation of $P \cup B$ which preserves P, B and I. By $Aut(\mathcal{S})$, we denote the full automorphism group of the GQ $\mathcal{S}$.

Two GQ's $\mathcal{S} = (P, B, I)$ and $\mathcal{S}' = (P', B', I')$ are said to be *isomorphic* if there are two bijective maps $\alpha : P \mapsto P'$ and $\beta : B \mapsto B'$ so that if pIL in $\mathcal{S}$, then $p^{\alpha} I' L^{\beta}$ in $\mathcal{S}'$; the pair (α, β) is called an *isomorphism* of $\mathcal{S}$ (on)to $\mathcal{S}'$ (or *between* $\mathcal{S}$ and $\mathcal{S}'$). If $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, then we write $\mathcal{S} \cong \mathcal{S}'$.

A *duality* of a generalized quadrangle $\mathcal{S}$ is an isomorphism between $\mathcal{S}$ and $\mathcal{S}^D$.

1.2.2 Symmetry

We will now give a second — for our purposes more convenient — definition for the notion "axis of symmetry". Suppose L is a line of a GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$. A *symmetry about* L is an automorphism of the GQ which fixes every line of $L^{\perp}$. The line L is called an *axis of symmetry* if there is a full group H of symmetries of size s about L. In such a case, if $M \in L^{\perp} \setminus \{L\}$, then H acts regularly on the points of M not incident with L. Dually, one defines the notions *symmetry about a point* and *center of symmetry.* It is easily seen that any axis of symmetry is a regular line.

Theorem 1.2.1 (FGQ, 8.1.2). *If a thick GQ $\mathcal{S}$ of order (s,t) has a non-identical symmetry θ about some line, then $st(s+1) \equiv 0 \mod s+t$.* ■

Theorem 1.2.2 (FGQ, 8.1.3). *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and suppose that L and M are distinct concurrent lines. If α is a symmetry about L and β a symmetry about M, then $\alpha\beta = \beta\alpha$.* ■

1.3 Subquadrangles of Generalized Quadrangles

A *subquadrangle*, or also *subGQ*, $\mathcal{S}' = (P', B', I')$ of a GQ $\mathcal{S} = (P, B, I)$ is a GQ for which $P' \subseteq P$, $B' \subseteq B$, and where I' is the restriction of I to $(P' \times B') \cup (B' \times P')$. If $\mathcal{S}' \neq \mathcal{S}$, then $\mathcal{S}'$ is called a *proper subquadrangle* of $\mathcal{S}$.

Theorem 1.3.1 (FGQ, 2.2.2). *Let $\mathcal{S}'$ be a proper subquadrangle of the GQ $\mathcal{S}$, where $\mathcal{S}$ has order (s,t) and $\mathcal{S}'$ has order (s,t') (so $t > t'$). Then we have*

(1) *$t \geq s$; if $s = t$, then $t' = 1$.*

(2) *If $s > 1$, then $t' \leq s$; if $t' = s \geq 2$, then $t = s^2$.*

(3) *If $s = 1$, then $1 \leq t' < t$ is the only restriction on t'.*

(4) *If $s > 1$ and $t' > 1$, then $\sqrt{s} \leq t' \leq s$ and $s^{3/2} \leq t \leq s^2$.*

(5) *If $t = s^{3/2} > 1$ and $t' > 1$, then $t' = \sqrt{s}$.*

(6) *Let $\mathcal{S}'$ have a proper subquadrangle $\mathcal{S}''$ of order (s,t''), $s > 1$. Then $t'' = 1$, $t' = s$ and $t = s^2$.* ■

Theorem 1.3.2 (FGQ, 2.3.1). *Let $\mathcal{S}' = (P', B', I')$ be a substructure of the GQ $\mathcal{S}$ of order (s,t) so that the following two conditions are satisfied:*

(i) *if $x, y \in P'$ are distinct points of $\mathcal{S}'$ and L is a line of $\mathcal{S}$ such that $xILIy$, then $L \in B'$;*

(ii) *each element of B' is incident with $s+1$ elements of P'.*

Then there are four possibilities:

(1) *$\mathcal{S}'$ is a dual grid, so $s = 1$;*

(2) *the elements of B' are lines which are incident with a distinguished point of P, and P' consists of those points of P which are incident with these lines;*

(3) *$B' = \emptyset$ and P' is a set of pairwise non-collinear points of P;*

(4) *$\mathcal{S}'$ is a subquadrangle of order (s,t').* ■

The following result is now easy to prove.

Theorem 1.3.3 (FGQ, 2.4.1). *Let θ be an automorphism of the GQ $\mathcal{S} = (P, B, I)$ of order (s,t). The substructure $\mathcal{S}_\theta = (P_\theta, B_\theta, I_\theta)$ of $\mathcal{S}$ which consists of the fixed elements of θ must be given by (at least) one of the following:*

(i) *$B_\theta = \emptyset$ and P_θ is a set of pairwise non-collinear points;*

(i)′ *$P_\theta = \emptyset$ and B_θ is a set of pairwise non-concurrent lines;*

(ii) *P_θ contains a point x so that $y \sim x$ for each $y \in P_\theta$, and each line of B_θ is incident with x;*

(ii)′ *B_θ contains a line L so that $M \sim L$ for each $M \in B_\theta$, and each point of P_θ is incident with L;*

(iii) *$\mathcal{S}_\theta$ is a grid;*

(iii)′ *$\mathcal{S}_\theta$ is a dual grid;*

(iv) *$\mathcal{S}_\theta$ is a subGQ of $\mathcal{S}$ of order (s', t'), $s', t' \geq 2$.* ■

A *whorl about a point* p of the GQ $\mathcal{S}$ is a collineation fixing p linewise. A *whorl about a line* L is a collineation fixing L pointwise.

Theorem 1.3.4 (FGQ, 8.1.1). *Let θ be a nontrivial whorl about p of the GQ $\mathcal{S} = (P, B, I)$ of order (s,t), $s \neq 1 \neq t$. Then one of the following must hold for the fixed element structure $\mathcal{S}_\theta = (P_\theta, B_\theta, I_\theta)$.*

(1) *$y^\theta \neq y$ for each $y \in P \setminus p^\perp$.*

(2) *There is a point y, $y \not\sim p$, for which $y^\theta = y$. Put $V = \{p, y\}^\perp$ and $U = V^\perp$. Then $V \cup \{p, y\} \subseteq P_\theta \subseteq V \cup U$, and $L \in B_\theta$ if and only if L joins a point of V with a point of $U \cap P_\theta$.*

(3) *$\mathcal{S}_\theta$ is a subGQ of order (s', t), where $2 \leq s' \leq s/t \leq t$, and hence $t < s$.* ■

The following lemma is an easy exercise in view of the results of this section. It will often be used without further notice.

Lemma 1.3.5. *Suppose $\mathcal{S}$ is a thick GQ of order (s,t) which contains an axis of symmetry L, and suppose $\mathcal{S}'$ is a thick proper subGQ of $\mathcal{S}$ of order (s, t') which contains L. Then L is also an axis of symmetry in $\mathcal{S}'$.* ■

1.4 Triads, 3-Regularity, Regularity and Property (G)

1.4.1 Triads and centers

A *triad* of points, respectively lines, is a triple of pairwise non-collinear points, respectively pairwise non-concurrent lines. Given a triad T, a *center* of T is just an element of $T^\perp$. A triad (of points or lines) is called *centric* if it has at least one center. It is called *unicentric* if there is precisely one center. The following result will often be utilized without further notice.

Theorem 1.4.1 (C. C. Bose and S. S. Shrikhande [14]). *Let $\mathcal{S}$ be a generalized quadrangle with parameters (s,t), $s \neq 1 \neq t$. Then $t = s^2$ if and only if the number of centers of each triad of points is a constant, and if this occurs, the constant is $s+1$. Dually, $s = t^2$ if and only if the number of centers of each triad of lines is a constant, and if this occurs, the constant is $t+1$.* ■

A proof of a generalization of Theorem 1.4.1 will be given in Appendix A (cf. Theorem A.4.2).

1.4.2 Antiregularity and regularity

The pair of points $\{x,y\}$, $x \not\sim y$, is *antiregular* if $|\{x,y\}^\perp \cap z^\perp| \leq 2$ for all $z \in P \setminus \{x,y\}$. The point x is *antiregular* if $\{x,y\}$ is antiregular for each $y \in P \setminus x^\perp$.

Theorem 1.4.2 (FGQ, 1.3.6). (i) *If $1 < s < t$, then $\{x,y\}$ neither is regular nor antiregular.*

(ii) *The pair $\{x,y\}$ is regular (with $s = 1$ or $s \geq t$) if and only if each triad $\{x,y,z\}$ has exactly $0, 1$ or $t+1$ centers. When $s = t$ this is if and only if each triad $\{x,y,z\}$ is centric.*

(iii) *If $s = t$, the pair $\{x,y\}$ is antiregular if and only if each triad $\{x,y,z\}$ has 0 or 2 centers.*

(iv) *If $s = t$ and each point of $x^\perp \setminus \{x\}$ is regular, then every point is regular.* ■

Let $\mathcal{S} = (P,B,I)$ be a GQ of order (s,t), $s > 1$ and $t > 1$.

Theorem 1.4.3 (FGQ, 1.5.1). (i) *If $\{x,y\}$ is antiregular with $s = t$, then s is odd.*

(ii) *If $\mathcal{S}$ has a regular point x and a regular pair $\{L,M\}$ of non-concurrent lines for which x is incident with no line of $\{L,M\}^{\perp\perp}$, then $s = t$ is even.*

(iii) *If x is coregular, then the number of centers of any triad $\{x,y,z\}$ has the same parity as $t+1$.*

(iv) *If each point is regular, then $t+1$ divides $(s^2-1)s^2$.* ■

In the next theorem, we always assume s and t to be different from 1.

Theorem 1.4.4 (FGQ, 1.5.2). (i) *If $\mathcal{S}$ has a regular point x and a regular line $L \not\mathrm{I} x$, then $s = t$ is even.*

(ii) *If $s = t$ is odd and if $\mathcal{S}$ contains two distinct regular points, then $\mathcal{S}$ is not self-dual.*

(iii) *If x is coregular and t is odd, then $\{x,y\}^{\perp\perp} = \{x,y\}$ for all $y \notin x^\perp$.*

(iv) *If x is coregular and $s = t$, then x is regular if and only if s is even.*

(v) *If x is coregular and $s = t$, then x is antiregular if and only if s is odd.* ■

1.4.3 3-Regularity, subGQ's and a useful theorem

Suppose $\mathcal{S}$ is a GQ of order (s, s^2), $s \neq 1$. Then for any triad of points $\{p, q, r\}$, $|\{p, q, r\}^{\perp}| = s+1$ by Theorem 1.4.1. Evidently $|\{p, q, r\}^{\perp\perp}| \leq s+1$. We say that $\{p, q, r\}$ is 3-*regular* provided that $|\{p, q, r\}^{\perp\perp}| = s+1$. The point p is 3-*regular* provided each triad containing p is.

Let $\{x, y, z\}$ be a 3-regular triad of the GQ $\mathcal{S} = (P, B, I)$ of order (s, s^2), $s \neq 1$ and s even. Let P' be the set of all points incident with lines of the form uv, with $u \in \{x, y, z\}^{\perp} = \mathbf{X}$ and $v \in \{x, y, z\}^{\perp\perp} = \mathbf{Y}$, and let B' be the set of lines L which are incident with at least two points of P'. Then J. A. Thas proves in [117] (see also 2.6.2 of FGQ) that, with I' the restriction of I to $(P' \times B') \cup (B' \times P')$, the geometry $\mathcal{S}' = (P', B', I')$ is a subGQ of $\mathcal{S}$ of order s. Moreover, $\{x, y\}$ is a regular pair of points of $\mathcal{S}'$, with $\{x, y\}^{\perp'} = \{x, y, z\}^{\perp}$ and $\{x, y\}^{\perp'\perp'} = \{x, y, z\}^{\perp\perp}$ (with the meaning of " $\perp'$ " being obvious).

We end this section with the following result of FGQ.

Theorem 1.4.5 (FGQ, 1.4.1). *Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, $m, n \geq 2$, be disjoint sets of pairwise non-collinear points of the GQ $\mathcal{S} = (P, B, I)$ of order (s, t), $s, t \neq 1$, and suppose that $X \subseteq Y^{\perp}$. Then $(m-1)(n-1) \leq s^2$. If equality holds, then one of the following must occur.*

(1) *$m = n = s+1$, and each point of $Z = P \setminus (X \cup Y)$ is collinear with precisely two points of $X \cup Y$.*

(2) *$m \neq n$. If $m < n$, then s is a divisor of t, $s < t$, $n = t+1$, $m = s^2/t+1$, and each point of $P \setminus X$ is collinear with either 1 or $t/s+1$ points of Y according as it is or is not collinear with some point of X.* ■

1.4.4 Property (G)

We end this section by defining "Property (G)", which is more general than 3-regularity.

Property (G). Let $\mathcal{S}$ be a generalized quadrangle of order (s, s^2), $s \neq 1$. Let x_1, y_1 be distinct collinear points. We say that the pair $\{x_1, y_1\}$ has *Property (G)*, or that $\mathcal{S}$ has *Property (G) at* $\{x_1, y_1\}$, if every triad $\{x_1, x_2, x_3\}$ of points for which $y_1 \in \{x_1, x_2, x_3\}^{\perp}$ is 3-regular. The GQ $\mathcal{S}$ has *Property (G) at the line* L, or the line L has *Property (G)*, if each pair of points $\{x, y\}$, $x \neq y$ and $xILIy$, has Property (G). If (x, L) is a flag, then we say that $\mathcal{S}$ has *Property (G) at* (x, L), or that (x, L) has *Property (G)*, if every pair $\{x, y\}$, $x \neq y$ and yIL, has Property (G).

Sometimes we will call the dual notion also "Property (G)".

Property (G) was introduced by S. E. Payne in his seminal paper [82], in connection with generalized quadrangles of order (q^2, q) arising from *flocks* of quadratic cones in $\mathbf{PG}(3, q)$, see Section 2.7.

1.5 The Classical and Dual Classical Generalized Quadrangles

1.5.1 The classical and dual classical generalized quadrangles

Consider a nonsingular quadric $\mathcal{Q}$ of Witt index 2, that is, of projective index 1, in $\mathbf{PG}(3,q)$, $\mathbf{PG}(4,q)$, $\mathbf{PG}(5,q)$, respectively. The points and lines of the quadric form a generalized quadrangle which is denoted by $\mathcal{Q}(3,q)$, $\mathcal{Q}(4,q)$, $\mathcal{Q}(5,q)$, respectively, and has order $(q,1)$, (q,q), (q,q^2), respectively. As $\mathcal{Q}(3,q)$ is a grid, its structure is trivial.

Recall that $\mathcal{Q}$ has the following canonical form:

(1) $X_0X_1 + X_2X_3 = 0$ if $d = 3$;

(2) $X_0^2 + X_1X_2 + X_3X_4 = 0$ if $d = 4$;

(3) $f(X_0, X_1) + X_2X_3 + X_4X_5 = 0$ if $d = 5$, where f is an irreducible binary quadratic form.

Next, let $\mathcal{H}$ be a nonsingular Hermitian variety in $\mathbf{PG}(3,q^2)$, respectively $\mathbf{PG}(4,q^2)$. The points and lines of $\mathcal{H}$ form a generalized quadrangle $H(3,q^2)$, respectively $H(4,q^2)$, which has order (q^2,q), respectively (q^2,q^3).

The variety $\mathcal{H}$ has the following canonical form:

$$X_0^{q+1} + X_1^{q+1} + \ldots + X_d^{q+1} = 0.$$

The points of $\mathbf{PG}(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity, form a GQ $W(q)$ of order q.

A symplectic polarity of $\mathbf{PG}(3,q)$ has the following canonical form:

$$X_0Y_1 - X_1Y_0 + X_2Y_3 - X_3Y_2.$$

The generalized quadrangles defined in this paragraph are the so-called 'classical generalized quadrangles', see Chapter 3 of FGQ. Their point-line duals are called the *dual classical generalized quadrangles*.

The following result will be used frequently without further reference.

Theorem 1.5.1 (FGQ, 3.2.1, 3.2.2 and 3.2.3). (i) *$\mathcal{Q}(4,q) \cong W(q)^D$;*

(ii) *$\mathcal{Q}(4,q) \cong W(q)$ if and only if q is even;*

(iii) *$\mathcal{Q}(5,q) \cong H(3,q^2)^D$.* ■

1.5.2 Combinatorial properties of the thick classical and dual classical generalized quadrangles

Suppose p and q are two non-collinear points of the GQ $\mathcal{S} = (P, B, I)$. Then we put

$$cl(p, q) = \{z \in \mathcal{S} \parallel z^{\perp} \cap \{p, q\}^{\perp\perp} \neq \emptyset\}.$$

A point x is *semiregular* provided that $r \in cl(p, q)$ whenever x is the unique center of $\{p, q, r\}$. A point x has *Property (H)* provided that $r \in cl(p, q)$ if and only if $p \in cl(q, r)$ whenever $\{p, q, r\}$ is a triad of points in $x^{\perp}$; we call the dual notion also "Property (H)". Each semiregular point clearly has Property (H).

Properties of $\mathcal{Q}(4, q)$. All lines are regular; all points are regular if and only if q is even; all points are antiregular if and only if q is odd; all points and lines are semiregular and have Property (H).

We also have the following important characterization theorem.

Theorem 1.5.2 (FGQ, 5.2.1). *A GQ of order s, $s > 1$, is isomorphic to $W(s)$ if and only if each point is regular.* ∎

Theorem 1.5.3 (FGQ, 5.2.6). *A GQ of order s, $s > 1$, is isomorphic to $W(s)$ if and only if it has a regular pair of non-concurrent lines $\{L, M\}$ with the property that any triad of points lying on lines of $\{L, M\}^{\perp}$ is centric.* ∎

Properties of $\mathcal{Q}(5, q)$. All lines are regular; all points are 3-regular; all points and lines are semiregular and have Property (H).

Theorem 1.5.4 (FGQ, 5.3.3). (i) *Let $\mathcal{S}$ be a GQ of order (s, s^2), $s > 1$ and s odd. Then $\mathcal{S} \cong \mathcal{Q}(5, s)$ if and only if $\mathcal{S}$ has a 3-regular point.*

(ii) *Let $\mathcal{S}$ be a GQ of order (s, s^2), s even. Then $\mathcal{S} \cong \mathcal{Q}(5, s)$ if and only if one of the following holds:*

(a) *all points of $\mathcal{S}$ are 3-regular;*

(b) *$\mathcal{S}$ has at least one 3-regular point not incident with some regular line.*

∎

Theorem 1.5.5 (FGQ, 5.3.5). (i) *A GQ $\mathcal{S}$ of order (s, t), $s > 1$, is isomorphic to $\mathcal{Q}(5, s)$ if and only if every centric triad of lines is contained in a proper subGQ of order (s, t').*

(ii) *A GQ $\mathcal{S}$ of order (s, t), $s > 1$ and $t > 1$, is isomorphic to $\mathcal{Q}(5, s)$ if and only if for each triad $\{u, v, w\}$ of points with distinct centers x, y the points u, v, w, x, y are contained in a proper subGQ of order (s, t').* ∎

Remark 1.5.6. Every line of the (thick) classical examples $\mathcal{Q}(4, q)$ and $\mathcal{Q}(5, q)$ is an axis of symmetry. This is easily seen since any line of $\mathcal{Q}(4, q)$, respectively $\mathcal{Q}(5, q)$, is regular and since every classical GQ is Moufang (cf. the introduction) — see, e.g., Theorem 2.3.4.

PROPERTIES OF $H(4,q^2)$. For each two distinct non-collinear points x, y we have that $|\{x,y\}^{\perp\perp}| = q+1$; if L and M are non-concurrent lines, then $\{L,M\}^{\perp\perp} = \{L,M\}$, but $\{L,M\}$ is not antiregular; all points and lines have Property (H) and all points but no lines are semiregular.

Theorem 1.5.7 (FGQ, 5.5.1). *A GQ of order (s^2,s^3), $s>1$, is isomorphic to the classical GQ $H(4,s^2)$ if and only if every hyperbolic line has at least $s+1$ points.* ■

1.6 Generalized Quadrangles with Small Parameters

Let $\mathcal{S}$ be a finite generalized quadrangle of order (s,t), $1<s\leq t$. We consider the cases $s=2,3,4$.

THE CASE $s=2$. If $s=2$, then $t\in\{2,4\}$. The GQ of order 2 is unique and is isomorphic to $\mathcal{Q}(4,2)$. The uniqueness of the GQ of order $(2,4)$ was proved independently at least five times, by S. Dixmier and F. Zara [30], H. Freudenthal [34], J. J. Seidel [106], E. E. Shult [108] and J. A. Thas [116].

THE CASE $s=3$. If $s=3$, then $t\in\{3,5,6,9\}$. The uniqueness of the GQ of order $(3,5)$ was proved by S. Dixmier and F. Zara in [30]. The uniqueness of the GQ of order $(3,9)$ was proved independently by S. Dixmier and F. Zara [30], and by P. J. Cameron in 1976 (see FGQ). For $s=t=3$ there are exactly two non-isomorphic GQ's, due independently to S. Dixmier and F. Zara [30] and S. E. Payne [71]. Finally, S. Dixmier and F. Zara proved in [30] that no GQ of order $(3,6)$ exists.

THE CASE $s=4$. If $s=4$, then $t\in\{4,6,8,11,12,16\}$. Nothing is known about the case $t=11$ or $t=12$. In the other cases, unique examples are known, but the uniqueness question is only settled for the case $t=4$. The proof is due to S. E. Payne [72, 73], with a gap filled by J. Tits in 1983, see also FGQ.

1.7 Hyperovals, Ovals and Ovoids in Projective Space

A *hyperoval* of a (finite) projective plane Π of order n is a set of $n+2$ points of Π no three of which are collinear. If a projective plane of order n admits a hyperoval, then n is even, see [46].
An *oval* of a (finite) projective plane Π of order n is a set of $n+1$ points of Π no three of which are collinear.
All ovals in a Desarguesian projective plane of odd order are known:

Theorem 1.7.1 (B. Segre [105]). *Each oval of* $\mathbf{PG}(2,q)$*, q odd, is an irreducible conic.* ■

A *translation oval $\mathcal{O}$ w.r.t. a point $p\in\mathcal{O}$* of $\mathbf{PG}(2,q)$ is an oval $\mathcal{O}$ such that there is a group H of automorphisms of $\mathbf{PG}(2,q)$ which stabilizes $\mathcal{O}$, which fixes the

unique tangent of $\mathcal{O}$ at p (in the usual sense) pointwise, and which acts regularly on $\mathcal{O} \setminus \{p\}$. Recently the theory of translation ovals was generalized by J. A. Thas and K. Thas to so-called 'translation generalized ovals' in $\mathbf{PG}(3n-1, q)$, see [135].

An *ovoid* $\mathcal{O}$ of $\mathbf{PG}(3, q)$, $q > 2$, is a set of $q^2 + 1$ distinct points no three of which are collinear; an *ovoid* of $\mathbf{PG}(3, 2)$ is a set of 5 points no four of which are coplanar. If $q > 2$, then an ovoid is a maximal sized set with its defining property.
The following theorem is (independently) due to A. Barlotti [8] and G. Panella [67].

Theorem 1.7.2 ([8]; [67]). *Each ovoid of* $\mathbf{PG}(3, q)$*, q odd, is an elliptic quadric.* ■

1.8 The $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$ of Tits

The first non-classical examples of generalized quadrangles were discovered by J. Tits, and appeared in the monograph of P. Dembowski [27].

Let $\mathcal{O}$ be an oval, respectively ovoid, in $\mathbf{PG}(d, q) = H$, for respectively $d = 2$ and $d = 3$. Embed $\mathbf{PG}(d, q)$ as a hyperplane in $\mathbf{PG}(d + 1, q) = H'$, and define a point-line incidence structure $T_d(\mathcal{O})$ as follows:

- The Points are of three types.
  - (i) A symbol (∞).
  - (ii) The hyperplanes Π of H' for which $|\Pi \cap \mathcal{O}| = 1$.
  - (iii) The points of $H' \setminus H$.
- The Lines are of two types.
  - (a) The points of $\mathcal{O}$.
  - (b) The lines of $H' \setminus H$ which intersect H in a point of $\mathcal{O}$.
- Incidence is defined as follows:
  - the point (∞) is incident with all the lines of Type (a) and with no other lines;
  - a point of Type (ii) is incident with the unique line of Type (a) contained in it and with all the lines of Type (b) which it contains (as subspaces);
  - a point of Type (iii) is incident with the lines of Type (b) that contain it.

Then J. Tits showed $T_d(\mathcal{O})$ to be a GQ of order q, respectively (q, q^2), for $d = 2$, respectively $d = 3$.

Theorem 1.8.1 (FGQ, 3.2.2 and 3.2.4). (i) *A $T_2(\mathcal{O})$ of order q is isomorphic to $\mathcal{Q}(4, q)$ if and only if $\mathcal{O}$ is an irreducible conic of* $\mathbf{PG}(2, q)$.

(ii) *A $T_3(\mathcal{O})$ of order (q, q^2) is isomorphic to $\mathcal{Q}(5, q)$ if and only if $\mathcal{O}$ is an elliptic quadric of* $\mathbf{PG}(3, q)$. ■

Remark 1.8.2. By Theorem 1.7.1 and Theorem 1.7.2, a $T_d(\mathcal{O})$ of order (q, q^{d-1}), $d = 2, 3$, is isomorphic to $\mathcal{Q}(d + 2, q)$ if q is odd.

We also have the following result, see Chapter 12 of FGQ.

Theorem 1.8.3 (FGQ, Chapter 12). *A $T_2(\mathcal{O})$ of Tits of order q is self-dual if and only if $\mathcal{O}$ is a translation oval of* $\mathbf{PG}(2, q)$ *(and then q is even).* ■

1.9 k-Arcs, Ovoids and Spreads in Generalized Quadrangles

1.9.1 k-Arcs, ovoids and spreads

A *k-arc* $\mathcal{K}$ of a GQ $\mathcal{S}$ is a set of k mutually non-collinear points. Then $k \leq st + 1$, see [91]. If $k = st + 1$, then $\mathcal{K}$ is an *ovoid* of $\mathcal{S}$. Dually, one defines *spreads*. In the obvious way, one defines *isomorphic ovoids* and *isomorphic spreads* of a GQ.

Theorem 1.9.1 (FGQ, 1.8.3). *A GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$ and $t > s^2 - s$, has no ovoid. Dually, a GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$ and $s > t^2 - t$, has no spread.* ■

Theorem 1.9.2 (FGQ, 1.8.4). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order s, $s > 1$, having a regular pair $\{x, y\}$ of non-collinear points. If $\mathcal{O}$ is an ovoid of $\mathcal{S}$, then $|\mathcal{O} \cap \{x, y\}^{\perp\perp}|$ and $|\mathcal{O} \cap \{x, y\}^{\perp}|$ are elements of $\{0, 2\}$, and $|\mathcal{O} \cap (\{x, y\}^{\perp} \cup \{x, y\}^{\perp\perp})| = 2$. If the GQ $\mathcal{S}$ of order s, $s > 1$, contains an ovoid and a regular point z not on $\mathcal{O}$, then s is even.* ■

Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, which contains a subGQ $\mathcal{S}'$ of order (s, t'), $t' > 1$, and let z be a point of $\mathcal{S} \setminus \mathcal{S}'$. Then by 2.2.1 of FGQ z is collinear with the points of an ovoid $\mathcal{O}_z$ of $\mathcal{S}'$. We say that $\mathcal{O}_z$ is 'subtended by z', and that $\mathcal{O}_z$ is a *subtended ovoid*. Suppose $t = s^2$, and let $\mathcal{S}'$ be a subGQ of order s. Suppose $\mathcal{O}_z$ is an ovoid of $\mathcal{S}'$ which is subtended by $z \in \mathcal{S} \setminus \mathcal{S}'$. Then by Theorem 1.4.5 there is at most one other point which also subtends $\mathcal{O}_z$ (since otherwise there would be spans of non-collinear points of size at least 3). If there is such a point, we say that $\mathcal{O}_z$ is *doubly subtended*. If for each point $z \in \mathcal{S} \setminus \mathcal{S}'$ the ovoid $\mathcal{O}_z$ is doubly subtended, we say that the subGQ $\mathcal{S}'$ is *doubly subtended*.

Suppose $\mathbf{T}$ is a spread of the GQ $\mathcal{S}$ of order (s, t), $s, t > 1$. Then $\mathbf{T}$ is *Hermitian* or *regular* or *normal* if for every two distinct lines L and M of $\mathbf{T}$, the pair $\{L, M\}$ is regular (so $|\{L, M\}^{\perp\perp}| = s + 1$) and $\{L, M\}^{\perp\perp} \subseteq \mathbf{T}$. Let $\mathbf{T}$ be a spread of $\mathcal{S}$. Then $\mathbf{T}$ is *locally Hermitian* or *semiregular* or *seminormal* w.r.t. the line L if for every line $M \neq L$ of $\mathbf{T}$, the pair $\{L, M\}$ is regular and $\{L, M\}^{\perp\perp} \subseteq \mathbf{T}$. Dually, we define *(locally) Hermitian ovoids*, *(semi)regular ovoids* and *(semi)normal ovoids*.

Finally, the following result of S. Ball, P. Govaerts and L. Storme [7] will be needed in the present work.

Theorem 1.9.3 ([7]). *Any ovoid of $\mathcal{Q}(4,p)$, p a prime, is an elliptic quadric.* ■

1.9.2 Complete arcs in generalized quadrangles

A k-arc is *complete* if it is not contained in a k'-arc with $k' > k$. The following theorem is an important observation.

Theorem 1.9.4 (FGQ, 2.7.1). *An $(st-m)$-arc in a GQ of order (s,t), where $-1 \leq m < t/s$ and $s \neq 1 \neq t$, is always contained in a uniquely defined ovoid.* ■

Hence it is a natural question to ask whether or not complete $(st-t/s)$-arcs exist. The following result answers the question almost completely in the case of the known GQ's (cf. Chapter 3).

Theorem 1.9.5 (K. Thas [144]). *Let $\mathcal{S}$ be a known GQ of order (s,t) with $s \neq 1 \neq t$, and suppose $\mathcal{S}$ has a complete $(st-t/s)$-arc $\mathcal{K}$. Then we necessarily have one of the following possibilities.*

(1) $\mathcal{S} \cong \mathcal{Q}(4,2)$ *and up to isomorphism there is a unique complete* 3*-arc.*

(2) $\mathcal{S} \cong \mathcal{Q}(5,2)$ *and up to isomorphism there is a unique complete* 6*-arc.*

(3) $\mathcal{S} \cong \mathcal{Q}(4,q)$ *with q odd.* ■

In a rather unexpected way, the existence/nonexistence question for complete $(st-t/s)$-arcs in GQ's of order (s,t) will turn up in this book.
It should be mentioned that much more general results than Theorem 1.9.5 are contained in [144], [156] and [149], including connections with other extremal combinatorial GQ problems (and solutions).

For $q = 3$, examples of complete (q^2-1)-arcs are known — see K. Thas [144]. In [144], we conjectured that no complete (q^2-1)-arcs could exist in $\mathcal{Q}(4,q)$, q odd, for $q > 3$. Very recently, T. Penttila [99] discovered complete (q^2-1)-arcs in $\mathcal{Q}(4,q)$ for $q = 5,7,11$, with the aid of a computer. Hence the following new version of the conjecture:

CONJECTURE. *If $\mathcal{S} \cong \mathcal{Q}(4,q)$ and q is odd, then $\mathcal{S}$ contains no complete (q^2-1)-arcs if q is sufficiently large.*

1.10 Some Permutation Group Theory

Let X be a set and let G be a group acting faithfully on X. Then the pair (X,G) is said to have *permutation rank* n, $n > 1$, if G acts transitively on X and if the stabilizer G_x of some element x of X in G has exactly n orbits in X. A rank 2

group is the same as a 2-*transitive group*. The group G acts n-*transitively*, $n \geq 2$, on X if G acts transitively on X and if G_x acts $(n-1)$-transitively on $X \setminus \{x\}$ for some $x \in X$.
The group G acts *semiregularly* on X if no nontrivial element of G fixes a point of X (in the finite case, there follows that $|G|$ divides $|X|$). The group G acts *regularly* on X if it acts semiregularly and transitively on X (in the finite case, this is equivalent to saying that $|X| = |G|$ and that G acts semiregularly on X).

Chapter 2

Elation Generalized Quadrangles, Translation Generalized Quadrangles and Flocks

We will generalize (with proofs) several theorems of Chapter 8 of *Finite Generalized Quadrangles*, so that the present chapter may serve as a general reference, but also as a necessary reference for the proof of the main result of this work.

The proofs of the results given in this chapter are taken from K. Thas [142].

2.1 Elation Generalized Quadrangles and Translation Generalized Quadrangles

Let $\mathcal{S} = (P, B, I)$ be a finite generalized quadrangle. An *elation about the point* $p \in P$ is a whorl about p that fixes no point of $P \setminus p^{\perp}$. Dually, one defines *elations about a line*. If θ is an elation about p, then we will often say that p is the *center* of θ. By definition, the identical permutation is an elation (about every point). If p is a point of the GQ $\mathcal{S}$ for which there exists a group of elations G about p which acts regularly on the points of $P \setminus p^{\perp}$, then $\mathcal{S}$ is said to be an *elation generalized quadrangle (EGQ)* with *base-point* or *elation point* p and *elation group* (or *base-group*) G, and we sometimes write $(\mathcal{S}^{(p)}, G)$ or $\mathcal{S}^{(p)}$ for $\mathcal{S}$. Dually, we define the *base-line* of an EGQ.

Remark 2.1.1. Usually, we only work with EGQ's that are thick. It will always be clear from the context in which results are stated when this is assumed, and therefore we will not bother each time to mention this.

Two basic problems in the theory of EGQ's are the following.

(1) *Given a thick EGQ* $(\mathcal{S}^{(x)}, G)$*, is the set of elations about* x *necessarily a group?*

(2) *Is each thick GQ an EGQ for some point or line?*

Each thick GQ of order $(s-1, s+1)$ is a counter example for (2) if $s > 3$. This is mentioned without a reference in Chapter 8 of FGQ, and contained in S. E. Payne and K. Thas [95] with a proof, in order to provide a reference on that observation. In S. E. Payne and K. Thas [94], it is shown that the *Kantor semifield flock GQ's* (cf. Section 3.4.1) of order (q^2, q), $q > 1$, provide a negative answer to Question (1) for its point (∞). The approach is constructive; they give an explicit example of a collineation θ which is an elation about (∞), so that $\langle\theta\rangle$ contains whorls which are not elations. Therefore, they introduce a *standard elation* as an elation (about a point) which generates a group of elations. Then they ask the following question:

(3) *Given a thick EGQ* $(\mathcal{S}^{(x)}, G)$*, is the set of standard elations about* x *necessarily a group?*

In [94], Question (3) is then answered (positively) for several large classes of GQ's (including the aforementioned Kantor semifield flock GQ's). Also, in [95], a criterion is developed to answer Question (1), and then the problem is solved for each known (class of) GQ('s). For some of the classical examples, these results were already obtained by K. Thas and H. Van Maldeghem in [158].

If a GQ $(\mathcal{S}^{(p)}, G)$ is an EGQ with elation point p, and if each line incident with p is an axis of symmetry, then we say that $\mathcal{S}$ is a *translation generalized quadrangle (TGQ)* with *base-point* p and *translation group* (or *base-group*) G. The elements of a translation group are called *translations*. More abstractly, we say that a GQ has a *translation point* p if each line incident with p is an axis of symmetry. Dually, we define *translation lines*. TGQ's were introduced by J. A. Thas in [115] for the case $s = t$, and by S. E. Payne and J. A. Thas in Chapter 8 of FGQ for the general case. For a general (recent) reference on TGQ's, we refer the reader to J. A. Thas and K. Thas [136].

Note. Recall Remark 2.1.1 when working with TGQ's.

Suppose (X, G) is a permutation group (where G acts on X) which satisfies the following properties:

(1) G acts transitively but not regularly on X;

(2) there is no nontrivial element of G with more than one fixed point in X.

Then (X, G) is a *Frobenius group* (or G is a *Frobenius group* in its action on X). Define $N \subseteq G$ by:

$$N = \{g \in G \parallel f(g) = 0\} \cup \{\mathbf{1}\},$$

where $f(g)$ is the number of fixed points of g in X. Then N is called the *Frobenius kernel* of G (or of (X, G)), and we have the following well known result.

Theorem 2.1.2 (Theorem of Frobenius). *N is a normal regular subgroup of G.* ■

Theorem 2.1.3 (FGQ, 8.2.4). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t) with $s \leq t$ and $s > 1$, and let p be a point for which $\{p, x\}^{\perp\perp} = \{p, x\}$ for all $x \in P \setminus p^{\perp}$. Let G be a group of whorls about p.*

(1) *If $y \sim p$, $y \neq p$, and if θ is a nonidentity whorl about p and y, then all points fixed by θ lie on py and all lines fixed by θ meet py.*

(2) *If θ is a nonidentity whorl about p, then θ fixes at most one point of $P \setminus p^{\perp}$.*

(3) *If G is generated by elations about p, then G is a group of elations, i.e. the set of elations about p is a group.*

(4) *If G acts transitively on $P \setminus p^{\perp}$ and $|G| > s^2 t$, then G is a Frobenius group on $P \setminus p^{\perp}$, so that the set of all elations about p is a normal subgroup of G of order $s^2 t$ acting regularly on $P \setminus p^{\perp}$, i.e. $\mathcal{S}^{(p)}$ is an EGQ with some normal subgroup of G as elation group.*

(5) *If G is transitive on $P \setminus p^{\perp}$ and G is generated by elations about p, then $(\mathcal{S}^{(p)}, G)$ is an EGQ.* ■

Theorem 2.1.4 (FGQ, 8.3.1). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s, t > 1$. Suppose each line through some point p is an axis of symmetry, and let G be the group generated by the symmetries about the lines through p. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.* ■

So *a thick GQ has a translation point x if and only if it is a TGQ with base-point x.*

Theorem 2.1.5 (FGQ, 8.2.3 and 8.5.2). *Suppose $(\mathcal{S}^{(x)}, G)$ is an EGQ of order (s, t), $s \neq 1 \neq t$. Then $(\mathcal{S}^{(x)}, G)$ is a TGQ if and only if G is an (elementary) abelian group. Also, in such a case there is a prime p and there are natural numbers n and k, where k is odd, such that either $s = t = p^n$ or $s = p^{nk}$ and $t = p^{n(k+1)}$. It follows that G is a p-group.* ■

2.2 4-Gonal Families and EGQ's

Suppose $(\mathcal{S}^{(p)}, G)$ is an EGQ of order (s, t), $s \neq 1 \neq t$, with elation point p and elation group G, and let q be a point of $P \setminus p^{\perp}$. Let $L_0, L_1, \ldots, L_t$ be the lines incident with p, and define r_i and M_i by $L_i I r_i I M_i I q$, $0 \leq i \leq t$. Put $H_i = \{\theta \in G \parallel M_i^{\theta} = M_i\}$ and $H_i^* = \{\theta \in G \parallel r_i^{\theta} = r_i\}$, and $\mathcal{J} = \{H_i \parallel 0 \leq i \leq t\}$. Then $|G| = s^2 t$ and $\mathcal{J}$ is a set of $t + 1$ subgroups of G, each of order s. Also, for each i, H_i^* is a subgroup of G of order st containing H_i as a subgroup. Moreover, the following two conditions are satisfied:

(K1) $H_i H_j \cap H_k = \{\mathbf{1}\}$ for distinct i, j and k;

(K2) $H_i^* \cap H_j = \{\mathbf{1}\}$ for distinct i and j.

Conversely, if G is a group of order s^2t and $\mathcal{J}$ (respectively $\mathcal{J}^*$) is a set of $t+1$ subgroups H_i (respectively H_i^*) of G of order s (respectively of order st), and if the Conditions (K1) and (K2) are satisfied, then the H_i^* are uniquely defined by the H_i (H_i^* is sometimes called the *tangent space* at H_i), and $(\mathcal{J}, \mathcal{J}^*)$ is said to be a 4-*gonal family of type* (s,t) *in* G. Sometimes we will also say that $\mathcal{J}$ is a 4-*gonal family of type* (s,t) *in* G if this seems convenient.

Let $(\mathcal{J}, \mathcal{J}^*)$ be a 4-gonal family of type (s,t) in the group G of order s^2t, $s \neq 1 \neq t$. Define an incidence structure $\mathcal{S}(G, \mathcal{J})$ as follows.

- Points of $\mathcal{S}(G, \mathcal{J})$ are of three kinds:
  - (i) elements of G;
  - (ii) right cosets H_i^*g, $g \in G$, $i \in \{0, 1, \dots, t\}$;
  - (iii) a symbol (∞).
- Lines are of two kinds:
  - (a) right cosets H_ig, $g \in G$, $i \in \{0, 1, \dots, t\}$;
  - (b) symbols $[H_i]$, $i \in \{0, 1, \dots, t\}$.
- Incidence. A point g of Type (i) is incident with each line H_ig, $0 \leq i \leq t$. A point H_i^*g of Type (ii) is incident with $[H_i]$ and with each line H_ih contained in H_i^*g. The point (∞) is incident with each line $[H_i]$ of Type (b). There are no further incidences.

It is straightforward to check that the incidence structure $\mathcal{S}(G, \mathcal{J})$ is a GQ of order (s,t). Moreover, if we start with an EGQ $(\mathcal{S}^{(p)}, G)$ to obtain the family $\mathcal{J}$ as above, then we have that

$$(\mathcal{S}^{(p)}, G) \cong \mathcal{S}(G, \mathcal{J}).$$

Proof. For any $h \in G$, define θ_h by $g^{\theta_h} = gh$, $(H_ig)^{\theta_h} = H_igh$, $(H_i^*g)^{\theta_h} = H_i^*gh$, $[H_i]^{\theta_h} = [H_i]$, $(\infty)^{\theta_h} = (\infty)$, with $g \in G$, $H_i \in \mathcal{J}$, $H_i^* \in \mathcal{J}^*$. Then θ_h is an automorphism of $\mathcal{S}(G, \mathcal{J})$ which fixes the point (∞) and all lines of Type (b). If

$$G' = \{\theta_h \parallel h \in G\},$$

then clearly $G' \cong G$ and G' acts regularly on the points of Type (i). The statement now easily follows. ■

Theorem 2.2.1. *A group of order s^2t admitting a 4-gonal family is an elation group for a suitable elation generalized quadrangle.* ■

These results were first noted by W. M. Kantor [52].

We now have the following interesting properties.
Let $(\mathcal{S}^{(p)}, G)$ be an EGQ, and define H_i, with $i = 0, 1, \dots, t$, and $\mathcal{J}$ as above.

Theorem 2.2.2 (FGQ, 8.2.2). *H_i is a group of symmetries about the line L_i if and only if $H_i \trianglelefteq G$ (and hence $\mathcal{S}^{(p)}$ is a TGQ if and only if $H_i \trianglelefteq G$ for each i), only if L_i is a regular line. The line L_i is regular if and only if $H_iH_j = H_jH_i$ for all $H_j \in \mathcal{J}$.* ■

2.3 Characterizations of Abstract TGQ's

Recall the following results:

Theorem 2.3.1 (X. Chen and D. Frohardt [21]). *Let G be a group of order s^2t admitting a 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ of type (s,t). If there exist two distinct members in $\mathcal{J}$ which are normal subgroups of G, then s and t are powers of the same prime number p and G is an elementary abelian p-group.* ■

Theorem 2.3.2 (D. Hachenberger [37]). *Let G be a group of order s^2t admitting a 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ of type (s,t). If G is a group of even order, and if there exists a member of $\mathcal{J}$ which is a normal subgroup of G, then s and t are powers of 2 and G is an elementary abelian 2-group.* ■

In geometrical terms, Theorem 2.3.1 reads as follows: "Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), $s,t \neq 1$, and suppose that there are at least two axes of symmetry L and M through the elation point x, such that *the full groups of symmetries about L and M are completely contained in G*. Then s and t are powers of the same prime number p and G is an elementary abelian p-group." Hence, by Theorem 2.1.5, $\mathcal{S}^{(x)}$ is a translation generalized quadrangle with translation group G. In geometrical terms, we have the following for Theorem 2.3.2: "Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), with $s,t \neq 1$ and s or t even, and suppose that there is at least one axis of symmetry L through the elation point x, such that *the full group of symmetries about L is completely contained in G*. Then s and t are powers of 2 and G is an elementary abelian 2-group." Thus, $\mathcal{S}^{(x)}$ is a translation generalized quadrangle with translation group G.

For the purpose of the present work, Theorems 2.3.1 and 2.3.2 are not satisfactory in that sense, that we want to get rid of the "contained in" condition in the geometrical setting. This will be done in this section.

For the case $s = t$, we have the following result of FGQ (see the appendix of Chapter 6 for a new proof).

Theorem 2.3.3 (FGQ, 11.3.5). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order s, with $s \neq 1$. Suppose that there are at least three axes of symmetry through a point p, and let G be the group generated by the symmetries about these lines. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.* ■

Let (p, L, p') be a panel of the GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$. A (p, L, p')-*collineation* is a whorl about p, L and p'. We present the following result as a lemma:

Lemma 2.3.4 (FGQ, 9.2.1). *Suppose (p, L, p') is a panel of the GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$, and let θ be a (p, L, p')-collineation. Then θ is a symmetry about L if L is regular.* ■

Corollary 2.3.5. *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, and suppose that L is a line which contains a panel (p, L, p') for which there is a full group of (p, L, p')-collineations of size s. Then L is an axis of symmetry if and only if L is regular.* ■

Lemma 2.3.6. *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, with a whorl θ about a point p, and assume that L and M are lines such that $M \sim LIp\not{I}M$, and for which $M^\theta = M$. Moreover, suppose that L is a regular line, and let q be a point on L, different from p and not on M. Then we have one of the following possibilities:*

(1) *q is not fixed by θ;*

(2) *q is fixed by θ, and then also every line through q.*

Proof. Suppose that q is fixed by θ, and consider a line QIq, $Q \neq L$. Also, consider an arbitrary point rIM and not on L, and the line $[r, Q] = proj_r Q$. Let NIp be such that $N \sim [r, Q]$, and let the intersection point be r'. Now consider the point r^θ. Since L is a regular line, there holds that the line $[r^\theta, Q]$ also intersects N, say in r''. It is clear that $(r')^\theta = r''$, and that $[r, Q]^\theta = [r^\theta, Q]$. Since the point q is fixed, it follows easily that the line Q also is fixed by θ, and hence the lemma follows. ■

Corollary 2.3.7. *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, and let p be a point, and L and M lines, such that $M \sim LIp\not{I}M$. Also, suppose that L is a regular line. Assume that θ is a whorl about p fixing M, and fixing a point q on L, different from p and not on M. Then every line through $r = L \cap M$ is fixed by θ.*

Proof. By Lemma 2.3.6, every line through q is fixed by θ. Since M is fixed, also r is fixed, and by the same lemma the proof follows. ■

Lemma 2.3.8. *Let $\mathcal{S}$ be a GQ of order (s, t), $s, t \neq 1$, and suppose $\theta \neq \mathbf{1}$ is a whorl about distinct collinear points p and q. If moreover, the line pq is regular, then θ is a symmetry about pq.*

Proof. Suppose M is an arbitrary line of $(pq)^\perp$ not through p or q. Suppose U and U' are distinct lines through p, both different from pq, and define the line V, respectively V', by being the unique line of $\{U, M\}^{\perp\perp}$, respectively $\{U', M\}^{\perp\perp}$, which is incident with q. Then $\{U, V\}^{\perp\perp} \cap \{U', V'\}^{\perp\perp} = \{M\}$, and $[\{U, V\}^{\perp\perp}]^\theta = \{U, V\}^{\perp\perp}$, respectively $[\{U', V'\}^{\perp\perp}]^\theta = \{U', V'\}^{\perp\perp}$. Hence $M^\theta = M$, and thus every line of $(pq)^\perp$ is fixed by θ. Hence θ is a symmetry about pq. ■

Lemma 2.3.9. *Suppose $\mathcal{S}$ is a GQ of order (s,t), with $s,t \neq 1$. Let p be a point of the GQ, and suppose that L and M are lines such that $M \sim L \mathrm{I} p \not\mathrm{I} M$, with L a regular line. Suppose that θ is a whorl about p fixing M, and suppose q is a point on L, different from p and not on M, which is also fixed by θ. Then θ is a symmetry about L.*

Proof. By Lemma 2.3.6, all the lines through q (and also through $L \cap M$) are fixed by θ. From Lemma 2.3.8, the result follows. ■

Now suppose that $\mathcal{S}$ is a GQ with parameters (s,t), $s,t \neq 1$. Also, assume that p is a point and $L \mathrm{I} p$ a regular line, and again that $M \sim L$ is a line not through p. Suppose θ is a whorl about p which fixes M, and such that $\langle\theta\rangle$ acts semiregularly on the points of M not on L. So $|\langle\theta\rangle|$ is a divisor of s. Consider an arbitrary nontrivial element ϕ of $\langle\theta\rangle$ of prime-power order, and consider the action of $\langle\phi\rangle$ on the points of $X = L \setminus [\{p\} \cup \{L \cap M\}]$. Since $|X| = s-1$ and because of the fact that $s-1$ and s are coprime, there follows immediately that there is a point $x \in X$ for which $x^{\langle\phi\rangle} = \{x\}$ (that is, every element of $\langle\phi\rangle$ fixes x). By Lemma 2.3.9, this implies that $\langle\phi\rangle$ is a group of symmetries about L. Since a finite group is generated by its elements of prime-power order and since ϕ was arbitrary, there follows that also $\langle\theta\rangle$ is a group of symmetries about L (the product of two symmetries about the same line is clearly again a symmetry about this line).

We now obtain

Theorem 2.3.10. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and suppose that L is a regular line through the point p. Let M be a line for which $L \sim M \not\mathrm{I} p$. If H is a group of whorls about p which fixes M and which acts transitively on the points of M which are not incident with L, then H contains a full group of symmetries about L of order s (i.e. L is an axis of symmetry).*

Proof. If $|H| = s$, then the statement follows directly from the preceding observation, hence suppose that $|H| > s$. Put $X = M \setminus \{L \cap M\}$. If θ is a nontrivial element of H which fixes at least two points of X, then by Theorem 1.3.4 there follows that $t < s$, a contradiction since L is a regular line. Hence the permutation group (X,H) satisfies the following conditions:

(i) H acts transitively on X;

(ii) for every $x \in X$, we have that $|H_x| \neq 1$, and

(iii) the only element of H which fixes at least two elements of X is the trivial element.

Thus (X,H) is a Frobenius group and hence by Theorem 2.1.2, H contains a normal subgroup N which acts regularly on X. By the observation preceding Theorem 2.3.10, this group is a full group of symmetries about L of size s. ■

Theorem 2.3.11. (1) *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t \neq 1$, and suppose that L is a regular line through the point p. If G is a group of whorls about p which acts transitively on the points of $P \setminus p^\perp$, then L is an axis of symmetry. In particular, suppose that $(\mathcal{S}^{(p)}, G)$ is an EGQ of order (s, t), $s, t \neq 1$, with elation point p and elation group G, and suppose that LIp is a regular line. Then L is an axis of symmetry.*

(2) *Suppose $\mathcal{S}$ is an EGQ of order (s, t), $s, t \neq 1$, with elation point p. Then LIp is a regular line if and only if L is an axis of symmetry.* ■

In Section 8.1 of FGQ, the following property is noted: *The set of all symmetries about some line L of a GQ $\mathcal{S}$ of order (s, t), $s, t \neq 1$, is always a group, and every symmetry about L is an elation about L and about every point on L. Hence the group has at most size s. Thus, if the group of symmetries about L hás size s, then this group is unique.*

We hence have the following theorem.

Theorem 2.3.12. *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t \neq 1$, and suppose G is a group of whorls about the point p which acts transitively on $P \setminus p^\perp$.*

(1) *If L is a regular line through p, then L is an axis of symmetry, and the full group of symmetries about L is completely contained in G.*

(2) *If LIp is an axis of symmetry, with G_L the full group of symmetries about L, then G_L is completely contained in G.*

In particular, the same statements hold for elation generalized quadrangles.

Proof. From the proof of Theorem 2.3.11, there follows that G contains a group G_L of size s of symmetries about L, and because of the remark preceding the theorem, this group is unique; Part (1) of the statement follows.
Part (2) follows directly from Part (1) and the fact that an axis of symmetry is a regular line. ■

We are ready to state the converse of Theorem 2.2.2:

Theorem 2.3.13. *Let $\mathcal{S} = (P, B, I)$ be an EGQ of order (s, t) with elation point p and where $s, t \neq 1$, and suppose q is a point of $P \setminus p^\perp$. Suppose $L_0, L_1, \ldots, L_t$ are the lines through p, and suppose M_i are lines such that $L_i \sim M_i I q$. Let H_i be the subgroup of the elation group G which fixes M_i for all i, and put $\mathcal{J} = \{H_0, H_1, \ldots, H_t\}$. Then H_i is a group of symmetries about the line L_i if and only if $H_i \trianglelefteq G$ (and hence $\mathcal{S}^{(p)}$ is a TGQ if and only if $H_i \trianglelefteq G$ for each i), if and only if L_i is a regular line. The line L_i is regular if and only if $H_i H_j = H_j H_i$ for all $H_j \in \mathcal{J}$.* ■

We can now give an alternative, more (incidence) geometrical definition of translation generalized quadrangles without the use of symmetries, abelian groups or

Galois geometries (cf. the next section), as follows. Suppose $\mathcal{S} = (P, B, I)$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with a point p so that there is a group of whorls about p that acts transitively on $P \setminus p^{\perp}$. Then $\mathcal{S}$ is a TGQ with translation point p if and only if every line through p is a regular line.

Remark 2.3.14. This result was proved by J. A. Thas for $s = t$ in [115], see also [91, 8.3.3].

Theorem 2.3.15. *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), $s,t \neq 1$. If there are two distinct regular lines through the point x, then s and t are powers of the same prime number p, G is an elementary abelian p-group, and hence $\mathcal{S}^{(x)}$ is a TGQ with translation group G.*

Proof. Suppose L and M are regular lines through the point x. Then by Theorem 2.3.12, we know that L and M are axes of symmetry for which the full groups of symmetries (of size s) are completely contained in G. Hence, by Theorem 2.3.1, the proof follows immediately. ■

Theorem 2.3.16. *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), $s,t \neq 1$. If there is a regular line through the point x, and G is a group of even order, then s and t are powers of 2, G is an elementary abelian 2-group, and hence $\mathcal{S}^{(x)}$ is a TGQ with translation group G.*

Proof. Suppose L is a regular line through the point x. Then by Theorem 2.3.12, L is an axis of symmetry for which the full group of symmetries about L is contained in G. Hence, by Theorem 2.3.2, the proof is complete. ■

2.4 $T(n,m,q)$'s and Translation Duals of TGQ's

In this paragraph, we introduce the notion of $T(n,m,q)$'s, which are natural generalizations of the $T_d(\mathcal{O})$ constructions of Tits, $d \in \{2,3\}$. Here, $n \neq 0$ and $m \neq 0$ are natural numbers.

Suppose $H = \mathbf{PG}(2n+m-1,q)$ is the finite projective $(2n+m-1)$-space over $\mathbf{GF}(q)$, and let H be embedded in a $\mathbf{PG}(2n+m,q)$, say H'. Now define a set $\mathcal{O} = \mathcal{O}(n,m,q)$ of subspaces as follows: $\mathcal{O}$ is a set of q^m+1 $(n-1)$-dimensional subspaces of H, denoted $\mathbf{PG}(n-1,q)^{(i)}$, so that

(i) every three generate a $\mathbf{PG}(3n-1,q)$;

(ii) for every $i = 0,1,\ldots,q^m$, there is a subspace $\mathbf{PG}(n+m-1,q)^{(i)}$ of H of dimension $n+m-1$, which contains $\mathbf{PG}(n-1,q)^{(i)}$ and which is disjoint from any $\mathbf{PG}(n-1,q)^{(j)}$ if $j \neq i$.

If $\mathcal{O}$ satisfies these conditions for $n = m$, then $\mathcal{O}$ is called a *pseudo-oval* or a *generalized oval* or an $[n-1]$*-oval* of $\mathbf{PG}(3n-1,q)$. A [0]-oval of $\mathbf{PG}(2,q)$ is just an oval of $\mathbf{PG}(2,q)$. For $n \neq m$, $\mathcal{O}(n,m,q)$ is called a *pseudo-ovoid* or a *generalized*

ovoid or an $[n-1]$-*ovoid* or an *egg* of $\mathbf{PG}(2n+m-1,q)$. A $[0]$-ovoid of $\mathbf{PG}(3,q)$ is just an ovoid of $\mathbf{PG}(3,q)$.
The spaces $\mathbf{PG}(n+m-1,q)^{(i)}$ are the *tangent spaces* of $\mathcal{O}(n,m,q)$ at $\mathbf{PG}(n-1,q)^{(i)}$, or just the *tangents*. Sometimes we will call an $\mathcal{O}(n,n,q)$ also an 'egg' or a 'generalized ovoid' for the sake of convenience.
Generalized ovals were introduced by J. A. Thas in [114], and generalized to generalized ovoids by S. E. Payne and J. A. Thas in FGQ, cf. Chapter 8.

Then S. E. Payne and J. A. Thas prove in [115, 91] that from any egg $\mathcal{O}(n,m,q)$ there arises a GQ $T(n,m,q) = T(\mathcal{O})$ which is a TGQ of order (q^n, q^m) for some special point (∞). This goes as follows.

- The Points are of three types.
  (1) A symbol (∞).
  (2) The subspaces $\mathbf{PG}(n+m,q)$ of H' which intersect H in a $\mathbf{PG}(n+m-1,q)^{(i)}$.
  (3) The points of $H' \setminus H$.
- The Lines are of two types.
  (a) The elements of the egg $\mathcal{O}(n,m,q)$.
  (b) The subspaces $\mathbf{PG}(n,q)$ of $\mathbf{PG}(2n+m,q)$ which intersect H in an element of the egg.
- Incidence is defined as follows: the point (∞) is incident with all the lines of Type (a) and with no other lines; a point of Type (2) is incident with the unique line of Type (a) contained in it and with all the lines of Type (b) which it contains (as subspaces); finally, a point of Type (3) is incident with the lines of Type (b) that contain it.

Conversely, any TGQ can be seen in this way (that is, as a $T(n,m,q)$ associated to an egg $\mathcal{O}(n,m,q)$ in $\mathbf{PG}(2n+m-1,q)$), and whence

Theorem 2.4.1. *The study of translation generalized quadrangles is equivalent to the study of generalized ovoids.* ■

We already mentioned that for a TGQ of order (s,t), there are natural q, k and n, where k is odd and q is a prime power, so that either $s=t=q^n$ or $s=q^{nk}$ and $t=q^{n(k+1)}$, and if q is even, then $k=1$, see 8.6.1 of FGQ. Each TGQ $\mathcal{S}$ of order (s,t) with translation point (∞) has a *kernel* $\mathbb{K}$, which is a field with a multiplicative group isomorphic to the group of all collineations of $\mathcal{S}$ fixing the point (∞) and any given point not collinear with (∞) linewise. So we have $|\mathbb{K}| \leq s$. The field $\mathbf{GF}(q)$ is a subfield of $\mathbb{K}$ if and only if $\mathcal{S}$ is of type $T(n,m,q)$.

We will often use the notation of this section without further notice.

Theorem 2.4.2 (FGQ, see the proof of 8.7.4(i)). *A TGQ of order s is isomorphic to a $T_2(\mathcal{O})$ of Tits with $\mathcal{O}$ an oval of $\mathbf{PG}(2, s)$ if and only if $|\mathbb{K}| = s$.* ■

Theorem 2.4.3 (FGQ, 8.7.4). *Let $(\mathcal{S}^{(p)}, G)$ be the TGQ arising from the generalized ovoid $\mathcal{O} = \mathcal{O}(n, 2n, q)$. Then $\mathcal{S}^{(p)} \cong T_3(\mathcal{O})$ for some ovoid $\mathcal{O}$ of $\mathbf{PG}(3, q^n)$, if and only if one of the following holds:*

(i) *for a fixed point y, $y \not\sim p$, the group of all whorls about p fixing y has order $q^n - 1$, that is, the kernel has size q^n;*

(ii) *for each point z of $\mathbf{PG}(4n - 1, q)$ not contained in an element of $\mathcal{O}$, the $q^n + 1$ tangent spaces containing z have exactly $(q^n - 1)/(q - 1)$ points in common;*

(iii) *each $\mathbf{PG}(3n - 1, q)$ containing at least three elements of $\mathcal{O}$ contains exactly $q^n + 1$ elements of $\mathcal{O}$.* ■

Theorem 2.4.4 (J. A. Thas and K. Thas [134]; See Chapter 14). *Suppose $\mathcal{S} = T(\mathcal{O})$ is a translation generalized quadrangle of order (q^n, q^m) with translation point (∞), and let $\mathbf{GF}(q)$ be a subfield of the kernel $\mathbf{GF}(q')$ of $T(\mathcal{O})$, where $\mathcal{O}$ either is a generalized ovoid ($n \neq m$) or a generalized oval ($n = m$) in $\mathbf{PG}(2n + m - 1, q) \subseteq \mathbf{PG}(2n + m, q)$. Then every automorphism of $\mathcal{S}$ which fixes (∞) is induced by an automorphism of $\mathbf{PG}(2n + m, q)$ which fixes $\mathcal{O}$, and conversely.* ■

The following easy-to-prove theorem is extremely important for TGQ theory.

Theorem 2.4.5 (FGQ, 8.7.2(iv)). *If $n \neq m$, then the $q^m + 1$ tangent spaces of $\mathcal{O}(n, m, q)$ form an egg $\mathcal{O}^*(n, m, q)$ in the dual space of $\mathbf{PG}(2n + m - 1, q)$.* ■

So in addition to $T(n, m, q)$ there arises a TGQ $T(\mathcal{O}^*)$, also denoted $T^*(n, m, q)$, or $T^*(\mathcal{O})$. The TGQ $T(\mathcal{O}^*)$ is called the *translation dual* of the TGQ $T(\mathcal{O})$. Now let $n = m$, with q odd. Then similar observations can be made, see e.g. [91, p. 182]. So also in this case there arises a *translation dual* of the TGQ $T(\mathcal{O})$. However, the only known TGQ for $s = t$ odd, is the classical GQ $\mathcal{Q}(4, s)$, which is isomorphic to its translation dual.

There are examples known of TGQ's of order (q^n, q^{2n}), q odd, which are not isomorphic to their translation dual; see Chapter 3 for all the known examples with this property.

Note. In the following, if $\mathcal{S} = T(\mathcal{O})$ is a TGQ for some translation point x, then by $\mathcal{S}^*$ we will sometimes denote the translation dual $T(\mathcal{O}^*)$ of $T(\mathcal{O})$ (if it exists).

2.5 Good Generalized Ovoids, Good TGQ's

A TGQ $T(\mathcal{O})$ with $t = s^2$, $s = q^n$, is called *good* at an element $\pi \in \mathcal{O}$ (or is *good* at the corresponding line through (∞) of the TGQ) if for every two distinct elements

π' and π'' of $\mathcal{O} \setminus \{\pi\}$ the $(3n-1)$-space $\pi\pi'\pi''$ contains exactly $q^n + 1$ elements of $\mathcal{O}$. We also say that π is a *good element* of $T(\mathcal{O})$ or $\mathcal{O}$, or that the corresponding line is a *good line* of $T(\mathcal{O})$, and that $\mathcal{O}$ is *good* at its element π. In that case, it is easy to see that $\pi\pi'\pi''$ is skew to the other elements.
The connection between TGQ's and Property (G) is given by the following.

Theorem 2.5.1 (J. A. Thas [121]). *If the TGQ $\mathcal{S}^{(\infty)}$ contains a good element π, then its translation dual satisfies Property (G) at the corresponding flag $((\infty)', \pi')$.* ■

We also have

Theorem 2.5.2 (J. A. Thas [121]). *Suppose $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a TGQ of order (s, s^2), $s > 1$, for which $\mathcal{O}$ is good at its element π. Then $\mathcal{S}$ contains $s^3 + s^2$ subGQ's of order s which contain the line π. For s odd these subGQ's are isomorphic to the classical GQ $\mathcal{Q}(4, s)$.* ■

Remark 2.5.3. The fact that $\mathcal{S}$ has $s^3 + s^2$ subGQ's of order s through π is easy to obtain (by definition). In fact, from an abstract point of view, one could define a good TGQ $\mathcal{S}^{(p)}$ (of order (s, s^2), $s > 1$) as *a TGQ for which there is a line L incident with the translation point p so that there are $s^3 + s^2$ distinct subGQ's of $\mathcal{S}^{(p)}$ of order s containing L.*
Hence it is a very important problem to classify those GQ's of order (s, s^2), $s > 1$, having a line L contained in $s^3 + s^2$ distinct subGQ's of order s — see the relevant conjectures (and results) in Chapter 11 on that matter.

Theorem 2.5.4 (J. A. Thas [130]). *Let $\mathcal{S}^{(p)} = T(\mathcal{O})$ be a TGQ of order (s, s^2), s even, such that $\mathcal{O}$ is good at its element $\pi I p$. If $\mathcal{S}$ contains at least one subGQ of order s which is isomorphic to the GQ $\mathcal{Q}(4, s)$ and which contains the line π, then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, s)$.* ■

Remark 2.5.5. Recent work of M. R. Brown and M. Lavrauw [18] implies that if $\mathcal{S} = \mathcal{S}^{(x)}$ is a TGQ of order (s, s^2), $s > 1$ and s even, with a subGQ $\mathcal{S}' \cong \mathcal{Q}(4, s)$ containing x, then $\mathcal{S} \cong \mathcal{Q}(5, s)$. We will not need their result here.

2.6 q-Clans and Generalized Quadrangles

Let $\mathbb{F} = \mathbf{GF}(q)$, q any prime power, and put $G = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. Define a binary operation on G by

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta\alpha'^T, \beta + \beta').$$

This makes G into a group whose center is $C = \{(\overline{0}, c, \overline{0}) \in G \parallel c \in \mathbb{F}\}$. (Here, $\overline{0} = (0, 0)$.)

Let $\mathcal{C} = \{A_u \parallel u \in \mathbb{F}\}$ be a set of q distinct upper triangular 2×2-matrices over $\mathbb{F}$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is *anisotropic* whenever $u \neq r$,

i.e., $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0,0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$. Let

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}, \qquad\qquad x_u, y_u, z_u, u \in \mathbb{F}.$$

For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{2.1}$$

is a non-square of $\mathbb{F}$ whenever $r, u \in \mathbb{F}$, $r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$y_u \neq y_r \text{ and } tr((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = \mathbf{1} \tag{2.2}$$

whenever $r, u \in \mathbb{F}$, $r \neq u$.

Now we can define a family of subgroups of G in the following way:

$$A(u) = \{(\alpha, \alpha A_u \alpha^T, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A(\infty) = \{(\overline{0}, 0, \beta) \in G \parallel \beta \in \mathbb{F}^2\}.$$

Then put $\mathcal{J} = \{A(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$ and $\mathcal{J}^* = \{A^*(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So

$$A^*(u) = \{(\alpha, c, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2, c \in \mathbb{F}\},\ u \in \mathbb{F},$$

and

$$A^*(\infty) = \{(\overline{0}, c, \beta) \parallel \beta \in \mathbb{F}^2, c \in \mathbb{F}\}.$$

With $G, A(u), A^*(u), \mathcal{J}$ and $\mathcal{J}^*$ as above, the following important theorem is a combination of results of S. E. Payne [74] and W. M. Kantor [52].

Theorem 2.6.1 ([74]; [52]). *The pair $(\mathcal{J}, \mathcal{J}^*)$ is a 4-gonal family for G if and only if $\mathcal{C}$ is a q-clan. Hence if $\mathcal{C}$ is a q-clan, then it defines a GQ of order (q^2, q).* ■

Now let

$$A = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \qquad \text{and} \qquad A' = \begin{pmatrix} x' & y' \\ w' & z' \end{pmatrix}$$

be 2×2-matrices over $\mathbb{F}$. Then we say that A and A' are *equivalent*, and we write $A \equiv A'$, provided $x = x'$, $z = z'$, and $y + w = y' + w'$. Then for arbitrary 2×2-matrices B and B' over $\mathbb{F}$, we have that $\alpha B \alpha^T = \alpha B' \alpha^T$ for all $\alpha \in \mathbb{F}^2$, if and only if $B \equiv B'$. Hence we can also define a q-clan as follows. A *q-clan* is a set

$$\mathcal{C} = \{A_u \equiv \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix} \parallel u \in \mathbb{F}\}$$

of q distinct (equivalence classes of) 2×2-matrices over $\mathbb{F}$ for which $A_u - A_r$ is anisotropic whenever $u \neq r$.

2.7 Some Flock Geometry (and Translation Planes)

2.7.1 Flocks, flock generalized quadrangles and q-clans

Let $\mathcal{F}$ be a *flock* of the quadratic cone $\mathcal{K}$ with vertex v of $\mathbf{PG}(3, q)$, that is, a partition of $\mathcal{K} \setminus \{v\}$ into q disjoint (irreducible) conics.
In his well known paper on flock geometry [120], J. A. Thas showed in an algebraic way that (2.1) and (2.2) are exactly the conditions for the planes

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

of $\mathbf{PG}(3, q)$ to define a flock of the quadratic cone $\mathcal{K}$ with equation $X_0 X_1 = X_2^2$. We have the following theorem.

Theorem 2.7.1 (J. A. Thas [120]). *To any flock $\mathcal{F}$ of the quadratic cone of $\mathbf{PG}(3, q)$ corresponds an EGQ $\mathcal{S}$ of order (q^2, q).* ■

In the rest of this work, we denote by $\mathcal{S}(\mathcal{F})$ the GQ of order (q^2, q) which arises from $\mathcal{F}$ as in Theorem 2.7.1, and such a GQ is called a *flock generalized quadrangle.*

Remark 2.7.2. (i) Note that a flock generalized quadrangle is always thick (by definition).

(ii) Two flocks $\mathcal{F}$ and $\mathcal{F}'$ of the quadratic cone $\mathcal{K}$ of $\mathbf{PG}(3, q)$ are called *isomorphic* or *equivalent* provided that there is an element of $\mathbf{P\Gamma L}(4, q)$ which fixes $\mathcal{K}$ and maps $\mathcal{F}$ to $\mathcal{F}'$.

The following theorem yields a truly important observation.

Theorem 2.7.3 (S. E. Payne [82]). *Any flock GQ satisfies Property (G) at its point (∞).* ■

The following theorem solves a long-standing open problem.

Theorem 2.7.4 (J. A. Thas [127]). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, s^2), $s > 1$, and assume that $\mathcal{S}$ satisfies Property (G) at the flag (x, L). If s is odd, then $\mathcal{S}$ is the point-line dual of a flock GQ.* ■

There is also a strong result for the even case, but we refer the reader to [127], as we do not need that result for this work.

A *skew translation generalized quadrangle (STGQ)* with *base-point* p is an EGQ with elation group G, such that p is a center of symmetry with the property that G contains the full group of symmetries about p.

The next result is well known.

Theorem 2.7.5. *The point (∞) of a flock GQ $\mathcal{S}(\mathcal{F})$ always is a center of symmetry and every flock GQ is a skew translation generalized quadrangle.*

Proof. Let $\mathcal{S}(\mathcal{F})$ be of order (q^2, q). Then the group $C = \{(\overline{0}, c, \overline{0}) \in G \parallel c \in \mathbb{F}\} \leq G$ induces a group of symmetries of size q about (∞). ■

2.7.2 Translation planes and semifield flock generalized quadrangles

Suppose Π is a finite projective plane of order n, $n \geq 2$, and let L be a line of Π. Then L is a *translation line* of Π if there is a group of collineations of Π fixing L pointwise and acting regularly on the points of Π not incident with L. If a projective plane contains a translation line L, then we say that it is a *translation plane*. Often, the affine plane defined by deleting L is also called a *translation plane*. For standard terminology on translation planes, see M. Kallaher [51].

Translation planes can be constructed from flocks by constructing an ovoid of the Klein quadric from a flock of the quadratic cone in $\mathbf{PG}(3, q)$. This ovoid corresponds to a line spread of $\mathbf{PG}(3, q)$ via the Klein correspondence, which in turn gives rise to a translation plane via the André/Bruck-Bose construction. This was independently observed in 1976 by both M. Walker [166] and J. A. Thas. Suppose $\mathcal{F}$ is a flock of the quadratic cone in $\mathbf{PG}(3, q)$, and let $\Pi(\mathcal{F})$ be the translation plane which arises from $\mathcal{F}$ as above. If $\Pi(\mathcal{F})$ is a *semifield plane* (see [46]), then $\mathcal{F}$ is called a *semifield flock*. If $\mathcal{F}$ is a semifield flock, then $\mathcal{S}(\mathcal{F})^D$ is a TGQ, see, e.g., [85], and conversely, if $\mathcal{S}(\mathcal{F})^D$ is a TGQ, then $\mathcal{F}$ is a semifield flock (up to *derivation*, cf. Section 2.7.4). Throughout, we say that a TGQ of order (q, q^2) *arises from a flock* if it is the point-line dual of a flock GQ of order (q^2, q). The translation dual of a TGQ which arises from a flock will sometimes be called a *semifield flock TGQ*.

2.7.3 Symplectic translation planes and spreads

Let $\mathbf{S}$ be a line spread of $\mathbf{PG}(3, q)$. Then $\mathbf{S}$ is *symplectic* if all lines are totally isotropic for some symplectic polarity of $\mathbf{PG}(3, q)$. The translation plane arising from $\mathbf{S}$ by the André/Bruck-Bose construction is called a *symplectic translation plane*.

2.7.4 Flocks and BLT-sets

Let q be an odd prime power, and let $\mathcal{F} = \{\mathcal{C}_1, \ldots, \mathcal{C}_q\}$ be a flock of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3, q)$ with vertex v. Let $\mathcal{K}$ be embedded in a nonsingular quadric $\mathcal{Q}$ of a $\mathbf{PG}(4, q)$ containing $\mathbf{PG}(3, q)$ as a hyperplane so that $\mathcal{K} = \mathbf{PG}(3, q) \cap \mathcal{Q}$. There are unique points $p_1, \ldots, p_q$ of $\mathcal{Q}$ for which $\mathcal{C}_i = v^\perp \cap p_i^\perp$, $1 \leq i \leq q$, where '$\perp$' is relative to $\mathcal{Q}$. Then the condition that $\mathcal{C}_1, \ldots, \mathcal{C}_q$ are disjoint is precisely the condition that $V = \{v, p_1, \ldots, p_q\}$ is a set of $q + 1$ points of $\mathcal{Q}$ such that for $1 \leq i < j \leq q$, $\{v, p_i, p_j\}$ is a triad of the GQ $\mathcal{Q}$ for which $\{v, p_i, p_j\}^\perp = \emptyset$. The main theorem of [5] is that given such a set V, it is also true that for each triple (p_i, p_j, p_k), $0 \leq i < j < k \leq q$ (where $p_0 = v$), no point of $\mathcal{Q}$ is collinear (in $\mathcal{Q}$) with all three of the points. It follows that each point of $\mathcal{Q} \setminus V$ is collinear with 0 or 2 points of V. Such a set V of $q+1$ points of $\mathcal{Q}$ is called a *BLT-set* of $\mathcal{Q}$. L. Bader, G. Lunardon and J. A. Thas have showed in [5] that by using the BLT-set V, the

flock $\mathcal{F}$ of the quadratic cone $\mathcal{K}$ may be interpreted as one of a set of $q+1$ flocks (also called a *BLT-set*) — recall that q is odd [5]. Each of these flocks (which are said to be 'derived' from the original one) corresponds to a line of the GQ $\mathcal{S}(\mathcal{F})$ through (∞); each of the q 'new' flocks is obtained by recoordinatizing the GQ $\mathcal{S}(\mathcal{F})$ so as to interchange the line $[A(\infty)]$ and some other line through (∞). It follows that two flocks of a BLT-set are projectively equivalent if and only if the corresponding pair of lines of $\mathcal{S}(\mathcal{F})$ is in the same orbit of the automorphism group of $\mathcal{S}(\mathcal{F})$ [93].

Finally, it should be noted that each of the flocks determined by a given BLT-set gives rise to the same generalized quadrangle; see [88].

Chapter 3

The Known Generalized Quadrangles

In this chapter, it is our purpose to provide a census of the known finite thick generalized quadrangles, sometimes accompanied by some further comments. Since this work is partly a geometrical study of certain automorphism groups of GQ's, in some cases the size of the automorphism group is given (mostly based on S. E. Payne [83]). We will spend considerably more detail to TGQ's which arise from flocks, and to their translation duals. We omit a discussion on 'sporadic' examples[1]; only the Penttila-Williams TGQ and its translation dual are considered in that context. We also obtain an explicit form for the Penttila-Williams TGQ.
No proofs will be given in this chapter: it is purely expository, and detailed references will be stated whenever relevant. The chapter mainly serves for two purposes: first of all, each known finite GQ (or more generally, each known construction of finite GQ's) will (have to) be considered in the course of the proof of the classification. Secondly, the chapter provides a complete account of all known finite GQ's at present, and can be used as such.

3.1 The Classical and Dual Classical Examples

The thick classical and dual classical examples are $H(4,q^2)$, $H(3,q^2)$, $W(q)$, $\mathcal{Q}(4,q)$, $\mathcal{Q}(5,q)$ and $H(4,q^2)^D$, q an arbitrary prime power. For more details, see Section 1.5.[2]

[1] A detailed account is contained in S. E. Payne [85].
[2] A good reference on the size of their respective automorphism groups is [164].

3.2 The $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$ of J. Tits

See Section 1.8. If $\mathcal{S}$ is isomorphic to a $T_d(\mathcal{O})$ with $q+1$ points on a line and if $\mathcal{S}$ is non-classical, $d = 2, 3$, then

$$Aut(\mathcal{S}) \cong \mathbf{P\Gamma L}(d+1, q)_{\mathcal{O}},$$

where $\mathcal{O} \subseteq \mathbf{PG}(2, q) \subseteq \mathbf{PG}(3, q)$ if $d = 2$, and $\mathcal{O} \subseteq \mathbf{PG}(3, q) \subseteq \mathbf{PG}(4, q)$ if $d = 3$ (cf. [134]).

3.3 Generalized Quadrangles of Order $(s-1, s+1)$ and $(s+1, s-1)$

For each prime power q, R. W. Ahrens and G. Szekeres [1] constructed GQ's of order $(q-1, q+1)$. For q even, these examples were found independently by M. Hall, Jr. [38]. S. E. Payne [68] found a construction method which included all these examples and which produced some additional ones for q even, see [69, 70]. These examples yield the only known cases of GQ's of order (s, t) with $s \neq 1$ and $t \neq 1$, in which s and t are not powers of the same prime.

3.3.1 The GQ's $AS(q)$ of order $(q-1, q+1)$ of R. W. Ahrens and G. Szekeres, q an odd prime power

Define a point-line incidence structure $AS(q) = (P, B, I)$, q an odd prime power, as follows.

- The POINTS of P are the points of the affine 3-space $\mathbf{AG}(3, q)$.
- The LINES of B are the following curves of $\mathbf{AG}(3, q)$:

  (i) $x = \sigma,\ y = a,\ z = b$,

  (ii) $x = a,\ y = \sigma,\ z = b$,

  (iii) $x = c\sigma^2 - b\sigma + a,\ y = -2c\sigma + b,\ z = \sigma$,

  where the parameter σ ranges over $\mathbf{GF}(q)$ and where a, b, c are arbitrary elements of $\mathbf{GF}(q)$.

- INCIDENCE is the natural one.

Then $AS(q)$ is a GQ of order $(q-1, q+1)$ [1].

3.3.2 The GQ's S^{-}_{xy} of order $(q+1, q-1)$, $q = 2^h$

Let $\mathcal{O}$ be a hyperoval in $\mathbf{PG}(2,q)$, $q = 2^h$, and let $\mathbf{PG}(2,q) = H$ be embedded as a plane in $\mathbf{PG}(3,q) = H'$. Let x and y be distinct points of $\mathcal{O}$. The following GQ S^{-}_{xy} of order $(q+1, q-1)$ can then be constructed [68, 77].

- The POINTS are of three types:
  - (i) the points of $H' \setminus H$;
  - (ii) the planes through x not containing y;
  - (iii) the planes through y not containing x.
- The LINES are just those lines of H' which are not contained in H and which meet $\mathcal{O} \setminus \{x, y\}$ (necessarily in a unique point).
- INCIDENCE is inherited from H'.

3.3.3 The GQ's $T_2^*(\mathcal{O})$ of order $(q-1, q+1)$, $q = 2^h$

Let $\mathcal{O}$ be a hyperoval in $\mathbf{PG}(2,q)$, $q = 2^h$, and let $\mathbf{PG}(2,q) = H$ be embedded as a plane in $\mathbf{PG}(3,q) = H'$. Define an incidence structure $T_2^*(\mathcal{O})$ by taking

- for POINTS just those points of $H' \setminus H$;
- for LINES just those lines of H' which are not contained in H and which meet $\mathcal{O}$ (necessarily in a unique point);
- for INCIDENCE just that inherited from H'.

Then $T_2^*(\mathcal{O})$ is a GQ of order $(q-1, q+1)$ [1, 38].

3.3.4 The GQ's $\mathcal{P}(\mathcal{S}, x)$ of S. E. Payne

In this section, we recall the beautiful combinatorial construction of S. E. Payne [68].

Let x be a regular point of the GQ $\mathcal{S} = (P, B, I)$ of order s, $s > 1$. Define an incidence structure $\mathcal{P}(\mathcal{S}, x) = \mathcal{S}' = (P', B', I')$ as follows:

- The POINT SET P' is the set $P \setminus x^{\perp}$.
- The LINES of B' are of two types:
  - the elements of Type (a) are the lines of B which are not incident with x;
  - the elements of Type (b) are the hyperbolic lines $\{x, y\}^{\perp\perp}$ where $y \not\sim x$.
- INCIDENCE I' is containment (regarding a line of $\mathcal{S}$ as a set of points).

Then $\mathcal{S}'$ is a GQ of order $(s-1, s+1)$.

3.4 TGQ's which Arise from Flocks (and their Translation Duals)

3.4.1 Kantor semifield flock generalized quadrangles

A flock is called *linear* if all the flock planes contain a common line. In that case, it can be shown that the flock GQ is classical, and conversely.

Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3, q)$, q odd. Then the q planes π_t with equation

$$tX_0 - mt^\sigma X_1 + X_3 = 0,$$

where $t \in \mathbf{GF}(q)$, m is a given non-square in $\mathbf{GF}(q)$ and σ a given automorphism of $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$; see [120]. All the planes π_t contain the exterior point $(0, 0, 1, 0)$ of $\mathcal{K}$. The flock is linear if and only if $\sigma = \mathbf{1}$. Conversely, every nonlinear flock $\mathcal{F}$ of $\mathcal{K}$ for which the planes of the q conics share a common point, is of the type just described, see [120].
The corresponding GQ $\mathcal{S}(\mathcal{F})$, which is a TGQ for some base-line, was first discovered by W. M. Kantor, and is called the *Kantor semifield (flock) generalized quadrangle*. The kernel $\mathbb{K}$ of the TGQ is the fixed field of σ, see [101]. The following was shown by S. E. Payne in [82].

Theorem 3.4.1 (S. E. Payne [82]). *Suppose a TGQ $\mathcal{S} = T(\mathcal{O})$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, $\mathcal{F}$ a Kantor semifield flock. Then $T(\mathcal{O})$ is isomorphic to its translation dual $T(\mathcal{O}^*)$.* ■

We also have the following, which is due to J. A. Thas and H. Van Maldeghem [138].

Theorem 3.4.2 (J. A. Thas and H. Van Maldeghem [138]). *Suppose that the TGQ $T(\mathcal{O})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ and q odd, is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, where the point (∞) of $\mathcal{S}(\mathcal{F})$ corresponds to the line η of Type (b) of $T(\mathcal{O})$. Then $T(\mathcal{O})$ is good at the element η if and only if $\mathcal{F}$ is a Kantor semifield flock.* ■

Note. In the rest of this work, we will often write 'Kantor (flock) GQ' instead of 'Kantor semifield (flock) GQ'. This cannot lead to confusion, as it is the only class of GQ's discovered by W. M. Kantor which we consider in the context of the classification.

The next interesting theorem classifies good TGQ's in the odd case.

Theorem 3.4.3 (A. Blokhuis, M. Lavrauw and S. Ball [13]). *Assume that $T(\mathcal{O})$ is a good TGQ of order (q^n, q^{2n}), q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ, with the additional condition that*

$$q \geq 4n^2 - 8n + 2.$$

Then $T(\mathcal{O})$ is isomorphic to the point-line dual of a Kantor flock GQ. ■

Let m be any non-square of $\mathbb{F} = \mathbf{GF}(q)$, q an odd prime power. For $t \in \mathbb{F}$, $\gamma \in \mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$, put

$$\hat{g}_t(\gamma) = \gamma \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \gamma^T + \left[\gamma \begin{pmatrix} 0 & 0 \\ 0 & -tm \end{pmatrix} \gamma^T \right]^{\sigma^{-1}}.$$

Put $G = \mathbb{F}^4 = \{(r,c,b,d) \parallel r,c,b,d \in \mathbb{F}\}$ with coordinatewise addition. Define subgroups in the following way: $B(\infty) = \{(r,0,0,0) \in G \parallel r \in \mathbb{F}\}$; $B^*(\infty) = \{(r,0,\gamma) \in G \parallel r \in \mathbb{F}, \gamma \in \mathbb{F}^2\}$. For $\gamma \in \mathbb{F}^2$, write $B(\gamma) = \{(-\hat{g}_c(\gamma), c, -c\gamma) \in G \parallel c \in \mathbb{F}\}$. Put $\gamma = (g_1, g_2) \in \mathbb{F}^2$. Then $B^*(\gamma) = \{(r,c,b,d) \in G \parallel c,b,d \in \mathbb{F}$ with $r = cg_1^2 + (cmg_2^2)^{\sigma^{-1}}\}$.
Then $\mathcal{J} = \{B(\gamma) \parallel \gamma \in \mathbb{F}^2 \cup \{\infty\}\}$ is a 4-gonal family for G with 'tangent space' $B^*(\gamma)$ at $B(\gamma)$. In the usual way we have a TGQ $\mathcal{S} = (\mathcal{S}^{(\infty)}, G) = \mathcal{S}(G,\mathcal{J})$ of order (q,q^2), and $\mathcal{S}$ is isomorphic to the (dual) Kantor GQ of order (q,q^2) (cf. [82]).

If $\mathcal{S} = \mathcal{S}(\mathcal{F})$ is a Kantor GQ, then $Aut(\mathcal{S})$ acts triply transitively on the lines through (∞) [83]. Suppose that $\mathcal{F}$ is nonlinear, and put $q = p^e$ for the prime p. If $\sigma^2 \neq \mathbf{1}$, then

$$|Aut(\mathcal{S})| = q^6(q+1)(q-1)^2 2e,$$

and if $\sigma^2 = \mathbf{1}$, $\sigma \neq \mathbf{1}$, then

$$|Aut(\mathcal{S})| = q^6(q+1)(q-1)^2 4e,$$

see S. E. Payne [83].

3.4.2 Roman generalized quadrangles

Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3,q)$, with $q = 3^r$ and $r > 2$. Then the q planes π_t with equation

$$tX_0 - (mt + m^{-1}t^9)X_1 + t^3X_2 + X_3 = 0,$$

$t \in \mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$ which is called the *Ganley flock*; see, e.g., [82]. The corresponding GQ $\mathcal{S}(\mathcal{F})$ is a TGQ for some base-line, and so the dual $\mathcal{S}(\mathcal{F})^D$ of $\mathcal{S}(\mathcal{F})$ is isomorphic to some $T(\mathcal{O})$. By [101], the kernel $\mathbb{K}$ is isomorphic to $\mathbf{GF}(3)$. S. E. Payne [82] shows the following.

Theorem 3.4.4 (S. E. Payne [82]). *$T(\mathcal{O})$ is not isomorphic to its translation dual $T(\mathcal{O}^*)$.* ∎

Further, S. E. Payne proves in [82] that $T(\mathcal{O}^*)$ is a TGQ which does not arise from a flock. In [82], the GQ's $T(\mathcal{O}^*)$ were called the *Roman generalized quadrangles*.

We present the Roman GQ's also under the form of their 4-gonal family.

Let n be any non-square of $\mathbb{F} = \mathbf{GF}(q)$, $q = 3^r$, $r > 2$. For $t \in \mathbb{F}$, $\gamma \in \mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$, put

$$\hat{g}_t(\gamma) = \gamma \begin{pmatrix} t & 0 \\ 0 & -nt \end{pmatrix} \gamma^T + \left[\gamma \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \gamma^T\right]^{1/3} + \left[\gamma \begin{pmatrix} 0 & 0 \\ 0 & -n^{-1}t \end{pmatrix} \gamma^T\right]^{1/9}.$$

Define $\hat{f} : \mathbb{F}^2 \times \mathbb{F}^2 \longrightarrow \mathbb{F}$ by

$$\hat{f}(\alpha, \gamma) = \alpha \begin{pmatrix} -1 & 0 \\ 0 & n \end{pmatrix} \gamma^T + \left[\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma^T\right]^{1/3} + \left[\alpha \begin{pmatrix} 0 & 0 \\ 0 & n^{-1} \end{pmatrix} \gamma^T\right]^{1/9}.$$

Then for all $d, t, u \in \mathbb{F}$ and $\alpha, \gamma \in \mathbb{F}^2$:

(1) $\hat{f}$ is biadditive and symmetric;

(2) $\hat{g}_t(\alpha + \gamma) - \hat{g}_t(\alpha) - \hat{g}_t(\gamma) = \hat{f}(t\alpha, \gamma) = \hat{f}(t\gamma, \alpha)$;

(3) $\hat{g}_{t+u}(\alpha) = \hat{g}_t(\alpha) + \hat{g}_u(\alpha)$;

(4) $\hat{g}_t(d\gamma) = \hat{g}_{td^2}(\gamma)$.

Put $G = \mathbb{F}^4 = \{(r, c, b, d) \parallel r, c, b, d \in \mathbb{F}\}$ with coordinatewise addition. Define subgroups in the following way: $B(\infty) = \{(r, 0, 0, 0) \in G \parallel r \in \mathbb{F}\}$; $B^*(\infty) = \{(r, 0, \gamma) \in G \parallel r \in \mathbb{F}, \gamma \in \mathbb{F}^2\}$. For $\gamma \in \mathbb{F}^2$, write $B(\gamma) = \{(-\hat{g}_c(\gamma), c, -c\gamma) \in G \parallel c \in \mathbb{F}\}$. Put $\gamma = (g_1, g_2) \in \mathbb{F}^2$. Then $B^*(\gamma) = \{(r, c, b, d) \in G \parallel c, b, d \in \mathbb{F}$ with $r = (cg_1^2 - cng_2^2 - bg_1 + dng_2) + (cg_1g_2 + bg_2 + dg_1)^{1/3} + (-cn^{-1}g_2^2 + dn^{-1}g_2)^{1/9}\}$. Then $\mathcal{J} = \{B(\gamma) \parallel \gamma \in \mathbb{F}^2 \cup \{\infty\}\}$ is a 4-gonal family for G with tangent space $B^*(\gamma)$ at $B(\gamma)$. So, in the usual way we have a TGQ $\mathcal{S} = (\mathcal{S}^{(\infty)}, G) = \mathcal{S}(G, \mathcal{J})$ of order (q, q^2), and $\mathcal{S}$ is isomorphic to the Roman GQ of order (q, q^2) (cf. [82]).

Recall that $q \geq 27$. Then $\mathcal{S}$ and $\mathcal{S}^*$ are non-classical, see [82], and by [83], we have that

$$|Aut(\mathcal{S}^*)| = q^6(q-1)2r$$

if $q > 27$. For $q = 27$, $|Aut(\mathcal{S}^*)| = q^6(q-1)8r$ (Maska Law, Private communication).

We emphasize that $\mathcal{S}^*$ is the point-line dual of the Ganley flock GQ.

3.4.3 The (sporadic) Penttila-Williams generalized quadrangle

In this paragraph, it is our goal to give an 'explicit' description of the *Penttila-Williams TGQ* of order $(3^5, 3^{10})$, as we did for the Roman TGQ's in the previous section. The method is taken from S. E. Payne [82].

Let $q = 3^5$. The q planes π_t with equation

$$tX_0 + 2t^9X_1 + t^{27}X_2 + X_3 = 0,$$

$t \in \mathbf{GF}(q)$, define a semifield flock of the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3, q)$. The flock, which is called the *Penttila-Williams flock*, was constructed by L. Bader, G. Lunardon and I. Pinneri in [4] using the *Penttila-Williams ovoid* of $\mathcal{Q}(4, 3^5)$ defined in [100]. The corresponding GQ, that is, the translation dual of $\mathcal{S}(\mathcal{F})^D$, is therefore referred to as the *(sporadic) Penttila-Williams generalized quadrangle*. The kernel of the Penttila-Williams GQ is isomorphic to $\mathbf{GF}(3)$ [85].

Define a 2×2-matrix A_t, $t \in \mathbf{GF}(q) = \mathbb{F}$, with $q = 3^5$, as

$$A_t = \begin{pmatrix} t & t^{27} \\ 0 & 2t^9 \end{pmatrix}.$$

Then with $A(t) = \{(\alpha, \alpha A_t \alpha^T, \alpha(A_t + A_t^T)) \parallel \alpha \in \mathbb{F}^2\}$ and $A(\infty) = \{(\overline{0}, 0, \beta) \parallel \beta \in \mathbb{F}^2\}$, $\mathcal{J} = \{A(\infty)\} \cup \{A(t) \parallel t \in \mathbb{F}\}$ is a 4-gonal family of type (q^2, q). This is the 4-gonal family which yields the TGQ $\mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ the Penttila-Williams flock. We now determine $(\mathcal{S}(\mathcal{F})^D)^*$.

Suppose that $G = \{(a, b, c, d) \parallel a, b, c, d \in \mathbb{F}\}$ is a group provided with coordinatewise addition. Put $tr(r_0 r + c_0 \hat{g}_r(\gamma)) = 0$ for all $r \in \mathbb{F}$, where $\hat{g}_r(\gamma) = \gamma A_r \gamma^T$, and where $c_0 \in \mathbb{F}$ is fixed. Then with $\gamma = (g_0, g_1)$, we have that $tr(r_0 r + c_0(g_0^2 r + g_0 g_1 r^{27} + 2g_1^2 r^9)) = tr(r(r_0 + c_0 g_0^2) + r^9(2g_1^2 c_0) + r^{27}(c_0 g_0 g_1)) = 0$ for all r if and only if $r_0 = -c_0 g_0^2 - (2g_1^2 c_0)^{1/9} - (c_0 g_0 g_1)^{1/27}$. Now put $\hat{A}(\infty) = \{(r, 0, 0, 0) \in G \parallel r \in \mathbb{F}\}$, and $\hat{A}(\gamma) = \{(-\hat{g}_c(\gamma), c, -c\gamma) \in G \parallel c \in \mathbb{F}\}$, where

$$\hat{g}_t(\gamma) = \gamma \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \gamma^T + \left[\gamma \begin{pmatrix} 0 & 0 \\ 0 & 2t \end{pmatrix} \gamma^T\right]^{1/9} + \left[\gamma \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \gamma^T\right]^{1/27}.$$

Then $\mathcal{J} = \{\hat{A}(\infty)\} \cup \{\hat{A}(\gamma) \parallel \gamma \in \mathbb{F}^2\}$ is a 4-gonal family such that the GQ $\mathcal{S}(G, \mathcal{J})$ is a TGQ of order (q, q^2) for which the translation dual is the point-line dual of the flock GQ $\mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ the Penttila-Williams flock.

3.5 The Other Known Flock GQ's of Order (q^2, q), q Odd

Throughout this section we assume that $\mathbb{K} = \mathbf{GF}(q)$, q an odd prime power.

Traditionally, the known examples of flock quadrangles have been given in terms of the associated q-clans. In [98] T. Penttila gave a new construction of some flock (for $q \equiv \pm 1 \mod 10$) as a BLT-set (which still lacks a satisfactory description as a q-clan). The newest infinite family for q odd has $q = 3^e$ and was discovered by M. Law and T. Penttila as a generalization of an example with $q = 27$ that was studied via computer (cf. [63]). That infinite family appears in [62]. Below we give a census of all these families along with some information about the associated flocks, and the automorphism group of the corresponding GQ.

As q is odd in this section, we give the q-clans as symmetric matrices of the form

$$A_t = \begin{pmatrix} t & f(t) \\ f(t) & g(t) \end{pmatrix}, \quad t \in \mathbb{K};\ f, g : \mathbb{K} \to \mathbb{K}.$$

(For the purpose of belonging to a q-clan, a matrix $A = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$ is completely determined by three quantities: x, u and $z+y$. For q odd, one may require that A is symmetric.)

The corresponding BLT-set $\mathcal{P}$ of the quadric

$$\mathcal{Q} : X_4X_0 + X_1X_3 + X_2^2 = 0$$

is then given by

$$\mathcal{P} = \{(1, t, f(t), -g(t), tg(t) - (f(t))^2)\} \cup \{(0, 0, 0, 0, 1)\}.$$

Recall that $\mathcal{C} = \{A_t \parallel t \in \mathbb{K}\}$ is a q-clan if and only if

$$-\det(A_s - A_t) = (f(s) - f(t))^2 - (s-t)(g(s) - g(t))$$

is a non-square of $\mathbb{K} = \mathbf{GF}(q)$ whenever $s \neq t$. As in S. E. Payne [85], we follow the presentation and naming conventions of the survey by N. Johnson and S. E. Payne [50]. In the case where a BLT-set gives rise to more than one flock, the family is named simply by concatenating the (initials of the) names given to the non-isomorphic flocks. For a given q-clan $\mathcal{C}$, respectively BLT-set $\mathcal{P}$, the associated generalized quadrangle is denoted $\mathcal{S}(\mathcal{C})$, respectively $\mathcal{S}(\mathcal{P})$.

From here on, the present chapter is based on S. E. Payne [85].

3.5.1 Classical: For all q

Let $x^2 + bx + c$ be irreducible over $\mathbb{K}$. Then

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{1}{2}bt \\ \frac{1}{2}bt & ct \end{pmatrix} \parallel t \in \mathbb{K} \right\}.$$

3.5.2 FTW: For $q \equiv -1 \mod 3$

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{3}{2}t^2 \\ \frac{3}{2}t^2 & 3t^3 \end{pmatrix} \parallel t \in \mathbb{K} \right\}.$$

We have that

$$|Aut(\mathcal{S})| = q^6(q+1)(q-1)^2e,$$

if $\mathcal{S}$ is non-classical [83] and $q = p^e$, p a prime, and $Aut(\mathcal{S})$ is triply transitive on the lines through (∞).

3.5.3 $\mathbf{K}_2$/JP: For $q \equiv \pm 2 \mod 5$

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{5}{2}t^3 \\ \frac{5}{2}t^3 & 5t^5 \end{pmatrix} \| \; t \in \mathbb{K} \right\}.$$

There holds that

$$|Aut(\mathcal{S})| = q^5(q-1)^2 2e,$$

if $\mathcal{S}$ is non-classical [83], $q = p^e$ for the prime p, and $Aut(\mathcal{S})$ has two orbits on the lines through (∞).

3.5.4 $\mathbf{K}_3$/BLT: For $q = 5^e$

Let k be a non-square of $\mathbb{K}$. Then

$$\mathcal{C} = \left\{ \begin{pmatrix} t & 3t^2 \\ 3t^2 & k^{-1}t(1+kt^2)^2 \end{pmatrix} \| \; t \in \mathbb{K} \right\}.$$

If $q > 5$, then $\mathcal{S}$ is non-classical, and by [83] it holds that

$$|Aut(\mathcal{S})| = q^6(q-1)2e.$$

3.5.5 Fi: For all odd $q \geq 5$

The Fisher flock has a q-clan representation (first discovered by J. A. Thas [120]; first given in explicit q-clan form in [81]) which is described in detail in [50]. However, it is rather involved and not as elegant as the BLT representation of T. Penttila in [98], which we will give here.

Let $\mathbb{K} = \mathbf{GF}(q) \subseteq \mathbf{GF}(q^2) = \mathbb{K}'$, q odd and $q \geq 5$. Let η be a primitive element of $\mathbb{K}'$, and put $\eta = \zeta^{q-1}$ (so η has multiplicative order $q+1$). Put $T(x) = x + \bar{x}$, where $\bar{x} = x^q$, and consider

$$V = \{(x, y, a) \| \; x, y \in \mathbb{K}', a \in \mathbb{K}\}$$

as a 5-dimensional vector space over $\mathbb{K}$. Define a map $\mathcal{Q} : V \to \mathbb{K}$ by

$$\mathcal{Q}(x, y, a) = x^{q+1} + y^{q+1} - a^2.$$

Then $\mathcal{Q}$ is a quadratic form on V of which the polar form f is given by

$$f((x, y, a), (z, w, b)) = T(x\bar{z}) + T(y\bar{w}) - 2ab.$$

The BLT-set $\mathcal{P}$ is now given by

$$\mathcal{P} = \{(\eta^{2j}, 0, 1) \| \; 1 \leq j \leq \frac{q+1}{2}\} \cup \{(0, \eta^{2j}, 1) \| \; 1 \leq j \leq \frac{q+1}{2}\}.$$

For non-classical Thas-Fisher GQ's, we have that

$$|Aut(\mathcal{S})| = q^5(q+1)^2(q-1)2e,$$

where $q = p^e$, p a prime; see [83].

3.5.6 Pe: For all $q \equiv \pm 1 \mod 10$

This example was discovered by T. Penttila [98]. With the same notation as in the preceding example, the BLT-set is given by

$$\mathcal{P} = \{(2\eta^{2j}, \eta^{3j}, \sqrt{5}) \in V \parallel 0 \leq j \leq q\}.$$

3.5.7 LP: For all $q = 3^e$

For $t \in \mathbb{K}$ with $q = 3^e$ and n a fixed non-square of $\mathbb{K}$, let

$$A_t = \begin{pmatrix} t & t^4 + nt^2 \\ t^4 + nt^2 & -n^{-1}t^9 + t^7 + n^2t^3 - n^3t \end{pmatrix}.$$

Then $\mathcal{C} = \{A_t \parallel t \in \mathbb{K}\}$ is a q-clan. This construction is due to M. Law and T. Penttila, see [62]. The full collineation group of $\mathcal{S}(\mathcal{C})$ is determined in [84], with some very interesting consequences for flocks. Let us recall some facts about those collineation groups (cf. [84]).

- The lines $[A(\infty)]$ and $[A(0)]$ each yield orbits of size one.
- For each $\sigma \in Aut(\mathbb{K})$ and each choice of ± 1 there is a collineation of $\mathcal{S}(\mathcal{C})$ mapping $[A(t)]$ to $[A(\bar{t})]$, where

$$\bar{t} = \pm n^{\frac{1-\sigma}{2}} t^\sigma.$$

- If the orbit of $[A(t)]$ has size h, then the stabilizer of $[A(t)]$ has size $\frac{2e}{h}$.

3.6 Infinite Families of Flocks with q Even

For q even, the connection between flocks, spreads, q-clans, generalized quadrangles of order (q^2, q), subquadrangles of order q and so-called 'herds of ovals' is given in detail in the survey [50]. As $q = 2^e$, not only is there a generalized quadrangle $\mathcal{S}(\mathcal{C})$ associated with the q-clan $\mathcal{C}$ as well as a spread of $\mathbf{PG}(3, q)$, but there is also a family of $q + 1$ subquadrangles of $\mathcal{S}(\mathcal{C})$ each having order q, and a herd of ovals in $\mathbf{PG}(2, q)$. For a detailed account on both latter objects, see S. E. Payne [85].

3.6.1 The classical case

The q-clan is

$$\mathcal{C} = \left\{ \begin{pmatrix} t^{\frac{1}{2}} & t^{\frac{1}{2}} \\ 0 & \kappa t^{\frac{1}{2}} \end{pmatrix} \parallel t \in \mathbb{K} \right\},$$

where $\kappa \in \mathbb{K}$ with $tr(\kappa) = \mathbf{1}$ is fixed.

3.6.2 The FTWKB-examples, $q = 2^e$, e odd

Suppose $q = 2^e$, e odd. The q-clan is given by

$$\mathcal{C} = \left\{ \begin{pmatrix} t^{\frac{1}{4}} & t^{\frac{1}{2}} \\ 0 & \kappa t^{\frac{3}{4}} \end{pmatrix} \| \ t \in \mathbb{K} \right\}.$$

The associated GQ's were discovered by W. M. Kantor [52] essentially via q-clans. These examples are non-classical if $e \geq 2$ [86].

3.6.3 The examples of S. E. Payne, $q = 2^e$, e odd

Suppose $q = 2^e$, e odd. The q-clan is

$$\mathcal{C} = \left\{ \begin{pmatrix} t^{\frac{1}{6}} & t^{\frac{1}{2}} \\ 0 & t^{\frac{5}{6}} \end{pmatrix} \| \ t \in \mathbb{K} \right\}.$$

The GQ is classical if and only if $q = 2$ and FTWKB if and only if $q = 8$.

3.6.4 The Subiaco and Adelaide geometries

The Subiaco examples were first given by W. E. Cherowitzo, T. Penttila, I. Pinneri and G. F. Royle [24] as q-clans. They exist for all $q = 2^e$ and were new for $q \geq 32$, except that certain of the smaller examples had been found by computer and the general construction was obtained in pieces. Since their construction is rather technical, since a rather complete review of them is in [50], and since the new construction of the Adelaide geometries is via a technique that gives both the Subiaco and the Adelaide (as well as the classical) examples, we give only this new version and refer the reader to [24] (see also [96]) for the original constructions. This new construction was discovered during an attempt to generalize the cyclic construction of [97]. However, the problem of giving a direct connection between the Adelaide construction and that in [97] seems to be open.

Let $\mathbb{K}' = \mathbf{GF}(q^2)$ be a quadratic extension of $\mathbb{K} = \mathbf{GF}(q)$, $q = 2^e$. Let $\mathbf{1} \neq \beta \in \mathbb{K}'$ satisfy $\beta^{q+1} = \mathbf{1}$, and put $T(x) = x + x^q$ for all $x \in \mathbb{K}'$. Let $a \in \mathbb{K}$ and $f, g : \mathbb{K} \to \mathbb{K}$ be defined by:

$$a = \frac{T(\beta^m)}{T(\beta)} + \frac{1}{T(\beta^m)} + 1;$$

$$f(t) = f_{m,\beta}(t) = \frac{T(\beta^m)(t+1)}{T(\beta)} + \frac{T((\beta t + \beta^q)^m)}{T(\beta)(t + T(\beta)t^{\frac{1}{2}} + 1)^{m-1}} + t^{\frac{1}{2}},$$

and

$$ag(t) = ag_{m,\beta}(t) = \frac{T(\beta^m)}{T(\beta)}t + \frac{T((\beta^2 t + 1)^m)}{T(\beta)T(\beta^m)(t + T(\beta)t^{\frac{1}{2}} + 1)^{m-1}} + \frac{1}{T(\beta^m)}t^{\frac{1}{2}}.$$

Put

$$\mathcal{C} = \mathcal{C}_{m,\beta} = \left\{ \begin{pmatrix} f(t) & t^{\frac{1}{2}} \\ 0 & ag(t) \end{pmatrix} \parallel t \in \mathbb{K} \right\}.$$

Then W. E. Cherowitzo, C. M. O'Keefe and T. Penttila [23] prove the following.

– If $m \equiv \pm 1 \mod q+1$, then $\mathcal{C}$ is the classical q-clan for all $q = 2^e$ and for all β.

– If $q = 2^e$ with e odd and $m \equiv \pm\frac{q}{2} \mod q+1$, then $\mathcal{C}$ is the example first found as a q-clan by W. M. Kantor and which gives the Fisher-Thas-Walker flock.

– If $q = 2^e$ with $m \equiv \pm 5 \mod q+1$, then $\mathcal{C}$ is the *Subiaco q-clan* for all β such that if λ is a primitive element of $\mathbb{K}$ and $\beta = \lambda^{k(q-1)}$, then $q+1$ does not divide km.

– If $q = 4^e > 4$ and $m \equiv \pm\frac{q-1}{3} \mod q+1$, then for all β, $\mathcal{C}$ is a q-clan called the *Adelaide q-clan*.

The Subiaco examples. Given $q = 2^e$, the q-clan is unique up to equivalence.

The Adelaide examples. These are constructed for $q = 2^e$ with e even. For the examples with $q = 4^k$, $k \leq 8$, that were studied by computer, the group acts transitively on the lines through (∞). S. E. Payne conjectured in [85] that this must be true in general. That conjecture was then proved to be true by W. E. Cherowitzo and S. E. Payne in [22].

Chapter 4

Substructures of Finite Nets

Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s,t \neq 1$, with a regular point. Then there is a net which arises from this regular point. In this chapter, we prove that if such a net has a proper subnet with the same degree as the net, then it must be an affine plane of order t. Also, this affine plane induces a proper subquadrangle of order t containing the regular point, and we necessarily have that $s = t^2$. This result has many applications.
The results of this chapter which are presented with proofs are taken from K. Thas [146].

4.1 Nets and Generalized Quadrangles with a Regular Point

A (finite) *net of order* $k(\geq 2)$ *and degree* $r(\geq 2)$ is an incidence structure $\mathcal{N} = (P, B, I)$ satisfying the following properties.

(1) Each point is incident with r lines and two distinct points are incident with at most one line.

(2) Each line is incident with k points and two distinct lines are incident with at most one point.

(3) If p is a point and L is a line not incident with p, then there is a unique line M incident with p and not concurrent with L.

A net of order k and degree r has k^2 points and kr lines, see e.g. [10, 7.1, 7.4 and 7.5].
Subnets and *automorphisms* of a net are defined in the usual sense.

The connection between generalized quadrangles and nets is given by the following theorem.

Theorem 4.1.1 (FGQ, 1.3.1). *Let p be a regular point of a GQ $\mathcal{S} = (P, B, I)$ of order (s, t), $s \neq 1 \neq t$. Then the incidence structure with point set $p^\perp \setminus \{p\}$, with line set the set of spans $\{q, r\}^{\perp\perp}$, where q and r are non-collinear points of $p^\perp \setminus \{p\}$, and with the natural incidence, is the dual of a net of order s and degree $t + 1$. If in particular $s = t$, there arises a dual affine plane of order s. Also, in the case $s = t$, the incidence structure π_p with point set $p^\perp$, with line set the set of spans $\{q, r\}^{\perp\perp}$, where q and r are different points in $p^\perp$, and with the natural incidence, is a projective plane of order s.* ■

Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, and assume that p is a regular point of $\mathcal{S}$. Then by $\mathcal{N}_p$ we will denote the net which is the point-line dual of the dual net corresponding to p (denoted $\mathcal{N}_p^*$), as described in Theorem 4.1.1. The notations $\mathcal{N}_L$ and $\mathcal{N}_L^*$ are used for the net and its dual arising from a regular line L. We will also often utilize the notation π_p (or Π_p) as in Theorem 4.1.1, or π_L (Π_L) when the plane arises from the regular line L.

4.2 Nets and Subquadrangles, and Applications

The following theorem is taken from [146] and implies that a net which arises from a regular point in a thick GQ cannot contain proper subnets of the same degree and different from an affine plane.

Theorem 4.2.1. *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t \neq 1$, with a regular point p. Let $\mathcal{N}_p$ be the net which arises from p, and suppose $\mathcal{N}'_p$ is a subnet of the same degree as $\mathcal{N}_p$. Then we have the following possibilities:*

(1) *$\mathcal{N}'_p$ coincides with $\mathcal{N}_p$;*

(2) *$\mathcal{N}'_p$ is an affine plane of order t and $s = t^2$; also, from $\mathcal{N}'_p$ there arises a proper subquadrangle of $\mathcal{S}$ of order t having p as a regular point.*

If, conversely, $\mathcal{S}$ has a proper subquadrangle containing the point p and of order (s', t) with $s' \neq 1$, then it is of order t, and hence $s = t^2$. Also, there arises a proper subnet of $\mathcal{N}_p$ which is an affine plane of order t.

Proof. First suppose that $\mathcal{S}$ contains a proper subquadrangle $\mathcal{S}'$ of order (s', t), $s', t \neq 1$, containing the point p. Then p is also regular in $\mathcal{S}'$ and since $s' \neq 1$, it follows that $s' \geq t$. By Theorem 1.3.1 this implies that $s' = t$ and that $s = t^2$. By Theorem 4.1.1, the net $\mathcal{N}'_p$ arising from the point p in $\mathcal{S}'$ is an affine plane of order t, and this net is clearly a subnet of the net which arises from the point p in $\mathcal{S}$.

Conversely, suppose that $\mathcal{N}_p$ is the net which arises from the regular point p in the GQ $\mathcal{S}$, and that it contains a proper subnet $\mathcal{N}'_p$ of the same degree. In the following, we identify points of the net with the corresponding spans of points in the GQ, and we use the same notation.

Suppose $P_1, P_2, \ldots, P_k$ are the points of $\mathcal{N}'_p$, define a point set P' of $\mathcal{S}$ as consisting of the points of $[\bigcup P_i] \cup [\bigcup P_i^{\perp}]$, and define B' as the set of all lines of $\mathcal{S}$ through a point of P'. Then it is not hard to check that the following properties are satisfied for the geometry $\mathcal{S}' = (P', B', I')$, with $I' = I \cap [(P' \times B') \cup (B' \times P')]$:

(1) any point of P' is incident with $t+1$ lines of B';

(2) if two lines of B' intersect in $\mathcal{S}$, then they also intersect in $\mathcal{S}'$.

Then by the dual of Theorem 1.3.2, $\mathcal{S}'$ is a proper subquadrangle of order (s', t), $s' \neq 1$, and analogously as in the beginning of the proof, we have that $s' = t$ and $s = t^2$. Also, the affine plane of order t which arises from the regular point p in this subquadrangle is the subnet $\mathcal{N}'_p$. ■

Corollary 4.2.2. *A net $\mathcal{N}$ which is attached to a regular point of a GQ contains no proper subnet of the same degree as $\mathcal{N}$, other than (possibly) an affine plane.* ■

Corollary 4.2.3. *Suppose p is a regular point of the GQ $\mathcal{S}$ of order (s,t), $s,t \neq 1$, and let $\mathcal{N}_p$ be the corresponding net. If $s \neq t^2$, then $\mathcal{N}_p$ contains no proper subnet of degree $t+1$.* ■

The following corollary tells us that nets which arise from a regular point of a GQ and which do not contain affine planes, are very 'irregular'.

Corollary 4.2.4. *Let p be a regular point of a GQ $\mathcal{S}$ of order (s,t), $s,t \neq 1$, and suppose $\mathcal{N}_p$ is the corresponding net. Moreover, suppose $s \neq t^2$. If u, v and w are distinct lines of $\mathcal{N}_p$ for which $w \not\sim u \sim v$, then these lines generate the whole net (under the taking of spans).*

Proof. Consider the points of $p^{\perp} \setminus \{p\}$ which correspond to the lines u, v, w of $\mathcal{N}_p$, and denote them respectively in the same way. Then by Theorem 1.3.2, u, v and w generate a (not necessarily proper) subGQ $\mathcal{S}'$ of $\mathcal{S}$ of order (s', t), where $s' > 1$. By Theorem 4.2.1 this implies that $\mathcal{S}' = \mathcal{S}$, since $s \neq t^2$. Hence u, v and w, as lines of $\mathcal{N}_p$, generate $\mathcal{N}_p$. ■

Lemma 4.2.5. *Suppose $\mathcal{S}$ is a GQ of order (s, s^2), $s \neq 1$, and suppose that $\mathcal{S}'$ and $\mathcal{S}''$ are two proper subquadrangles of $\mathcal{S}$ of order s. Then one of the following possibilities occurs:*

(1) *$\mathcal{S}' \cap \mathcal{S}''$ is a set of s^2+1 pairwise non-collinear points (i.e. an ovoid) of $\mathcal{S}'$ and $\mathcal{S}''$;*

(2) *$\mathcal{S}' \cap \mathcal{S}''$ consists of a point p of $\mathcal{S}'$ (and $\mathcal{S}''$), together with all lines of $\mathcal{S}'$ (and $\mathcal{S}''$) through this point, and all points of $\mathcal{S}'$ (and $\mathcal{S}''$) incident with these lines;*

(3) *$\mathcal{S}' \cap \mathcal{S}''$ is a GQ of order $(s, 1)$;*

(4) *$\mathcal{S}' = \mathcal{S}''$.*

Proof. Every line of a GQ of order (s, s^2), $s \neq 1$, intersects any subGQ of order s. The proof now follows from Theorem 1.3.2 and a simple counting argument. ■

Theorem 4.2.6. *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s,t \neq 1$, and suppose ϕ is a nontrivial whorl about a regular point p. Also, suppose ϕ fixes distinct points q, r and u of $p^\perp \setminus \{p\}$ for which $q \sim r$ and $q \not\sim u$. Then we have one of the following possibilities.*

(1) *We have that $s = t^2$ and $\mathcal{S}$ contains a proper subquadrangle $\mathcal{S}'$ of order t. Moreover, if ϕ is not an elation, then $\mathcal{S}'$ is fixed pointwise by ϕ.*

(2) *ϕ is a nontrivial symmetry about p.*

Proof. It is clear that if v and w are non-collinear points of $p^\perp$ which are fixed by a whorl about p, then every point of the span $\{v,w\}^{\perp\perp}$ is also fixed by the whorl. Now suppose $\mathcal{N}_p$ is the net which arises from p, and suppose that $\mathcal{N}'_p$ is the (not necessarily proper) subnet of $\mathcal{N}_p$ of order $t+1$ which is generated by u, q and r. Then every point of $\mathcal{N}'_p$ is fixed by ϕ by the previous observation. If $\mathcal{N}'_p$ is proper, then by Theorem 4.2.1 it is an affine plane of order t and $s = t^2$. Also, there arises a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order t. If ϕ is not an elation, then by Theorem 1.3.4 it follows that there is a proper subquadrangle $\mathcal{S}_\phi$ of order (s', t), $s' \neq 1$, which is fixed pointwise (and then also linewise) by ϕ. Since $\mathcal{S}_\phi$ has a regular point, we have that $s' \geq t$. By Theorem 1.3.1, $\mathcal{S}'$ is necessarily of order t. From Lemma 4.2.5 now follows that $\mathcal{S}_\phi = \mathcal{S}'$.

If $\mathcal{N}'_p = \mathcal{N}_p$, then every point of $p^\perp$ is fixed by ϕ. Since ϕ is not the identity, it follows from Theorem 1.3.4 that ϕ is an elation and hence a symmetry about p.■

Remark 4.2.7. It may be clear to the reader that if ϕ and $\mathcal{S}' = \mathcal{S}_\phi$ are as in (1) of Theorem 4.2.6, then ϕ is an involution which acts semiregularly on the points and lines of $\mathcal{S} \setminus \mathcal{S}_\phi$. Hence $\mathcal{S}_\phi$ is a doubly subtended subGQ of $\mathcal{S}$.

Notation. By $gcd(n,m)$, with $m,n \in \mathbb{N}$, we denote the greatest common divisor of m and n. Sometimes, if it cannot create any confusion, we will also use the notation (n,m).

Theorem 4.2.8. *Let $(\mathcal{S}^{(p)}, G)$ be an EGQ of order (s,t), $s,t \neq 1$, and suppose p is a regular point. Moreover, suppose that $gcd(s-1,t) = 1$. Then we have one of the following:*

(1) *there is a full group of order t of symmetries about p which is completely contained in G, and hence $(\mathcal{S}^{(p)}, G)$ is a skew translation generalized quadrangle;*

(2) *$\mathcal{S}$ contains a proper subGQ of order t (for which p is a regular point), and consequently $s = t^2$.*

Proof. Suppose $\mathcal{S} = (P, B, I) = (\mathcal{S}^{(p)}, G)$ satisfies the desired properties, and assume that $(\mathcal{S}^{(p)}, G)$ is not a skew translation generalized quadrangle (note that

since the GQ contains a regular point, we know that $t \leq s$). Then there is no full group of symmetries of size t about p which is completely contained in the elation group G. Consider an arbitrary point q of $P \setminus p^{\perp}$; then the subgroup H of G which fixes every element of $\{q, p\}^{\perp}$ has order t, and acts regularly on $\{q, p\}^{\perp\perp} \setminus \{p\}$. Now consider an arbitrary but nontrivial element θ of H of prime-power order, say h. Then h is a divisor of t. Next, suppose L is an arbitrary line through p, and suppose X is the set of $s-1$ points on L which are different from p and not contained in $q^{\perp}$. Then it is clear that θ, and also every element of $\langle\theta\rangle$, has at least one fixed point u in X, as $gcd(s-1, t) = 1$. By the proof of Theorem 4.2.1, the points of $\{p, q\}^{\perp}$ together with u generate — as lines of $\mathcal{N}_p$ — a subnet $\mathcal{N}_\theta$ of $\mathcal{N}_p$ of the same degree as $\mathcal{N}_p$, and every point of $p^{\perp}$ which corresponds to a line of $\mathcal{N}_\theta$ is fixed by θ. Thus, if $\mathcal{N}_\theta = \mathcal{N}_p$, then θ is a nontrivial symmetry about p. Suppose that for *every* such $\theta \in H$ of prime-power order, the net $\mathcal{N}_\theta$ coincides with $\mathcal{N}_p$; then it is clear that H is a full group of symmetries about p of order t, since a finite group is always generated by its elements of prime-power order (a product of symmetries about the same point still gives a symmetry about this point). Moreover, H is completely contained in G, a contradiction since this would imply that $(\mathcal{S}^{(p)}, G)$ is a skew translation GQ.
Hence $\mathcal{N}_p$ has a proper subnet $\mathcal{N}_{\theta'}$ for some $\theta' \in H$ of prime-power order, and by Theorem 4.2.1, $\mathcal{N}_{\theta'}$ is an affine plane of order t. Also, $s = t^2$ and there arises a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order t which contains the point p. It is clear that p is also regular in $\mathcal{S}'$. Thus the proof is complete. ■

Remark 4.2.9. The collineation θ' induces a nontrivial symmetry about p in $\mathcal{S}'$.

Corollary 4.2.10. *Suppose $(\mathcal{S}^{(p)}, G)$ is an EGQ of order s, $s > 1$, and suppose that the elation point p is regular. Then there is a full group of symmetries $\mathcal{C}$ about p and $\mathcal{C}$ is completely contained in G, hence $\mathcal{S}^{(p)}$ is a skew translation generalized quadrangle with base-point p.* ■

4.3 Generalized Quadrangles with a Regular Point and Translation Nets

The following theorem shows that skew translation generalized quadrangles provide special nets. Recall that a *translation net* $\mathcal{N}$ is a net for which there is a group G of automorphisms of $\mathcal{N}$ each element of which fixes each parallel class of $\mathcal{N}$, and so that G acts regularly on the points of $\mathcal{N}$. If $\mathcal{N}$ has order k and degree r; $k, r \geq 2$, and if $k = r - 1$, then $\mathcal{N}$ is a translation plane.

Theorem 4.3.1. *Suppose $(\mathcal{S}^{(p)}, G)$ is a skew translation generalized quadrangle of order (s, t), $s, t \neq 1$, with elation point p. Then the net $\mathcal{N}_p$ is a translation net.*

Proof. Suppose $\mathcal{C}$ is the full group of order t of symmetries about p. If we consider the action of G on the net $\mathcal{N}_p$, then it is clear that $\mathcal{C}$ is precisely the kernel of this

action, hence the group $G/\mathcal{C}$ acts as a faithful automorphism group on $\mathcal{N}_p$. Also, it is clear that $G/\mathcal{C}$ acts semiregularly on the points of $\mathcal{N}_p$ ($\mathcal{C}$ is the only subgroup of G which contains nontrivial elements fixing a point of $\mathcal{N}_p$). Moreover, the group $G/\mathcal{C}$ has size s^2, and hence $G/\mathcal{C}$ acts regularly on the points of $\mathcal{N}_p$. Thus, since every parallel class of $\mathcal{N}_p$ is fixed by $G/\mathcal{C}$, it follows that $\mathcal{N}_p$ is a translation net.■

Remark 4.3.2. If we put $s = t$ in Theorem 4.3.1, then $\mathcal{N}_p$ is a translation plane, s is the power of a prime and $G/\mathcal{C}$ is elementary abelian; see e.g. [46, p. 100].

4.4 The Axiom of Veblen, $\mathcal{P}$-Nets and Generalized Quadrangles

We now introduce the *Axiom of Veblen* for dual nets.

Axiom of Veblen. If $L_1IxIL_2, L_1 \neq L_2, M_1 \not{I} x \not{I} M_2$, and if the line L_i is concurrent with the line M_j for all $i, j \in \{1, 2\}$, then M_1 is concurrent with M_2.

An example of a dual net $\mathcal{N}^*$ which is not a dual affine plane and which satisfies the Axiom of Veblen is the dual net H_q^n, $n > 2$, which is constructed as follows:

- the POINTS of H_q^n are the points of $\mathbf{PG}(n, q)$ not in a given subspace $\mathbf{PG}(n-2, q) \subset \mathbf{PG}(n, q)$;
- the LINES of H_q^n are the lines of $\mathbf{PG}(n, q)$ which have no point in common with $\mathbf{PG}(n-2, q)$;
- the INCIDENCE in H_q^n is the natural one.

By the following theorem, the dual nets H_q^n are characterized by the Axiom of Veblen.

Theorem 4.4.1 (J. A. Thas and F. De Clerck [132]). *Let $\mathcal{N}^*$ be a dual net with $s+1$ points on any line and $t+1$ lines through any point, where $t+1 > s$. If $\mathcal{N}^*$ satisfies the Axiom of Veblen, then $\mathcal{N}^* \cong H_q^n$ with $n > 2$ (hence $s = q$ and $t + 1 = q^{n-1}$).* ■

Let $\mathcal{N} = (\overline{P}, \overline{B}, \overline{I})$ be a net of order k and degree r. Further, let R be a line of $\mathcal{N}$ and let $\mathcal{P}$ be the parallel class of $\overline{B}$ containing R, that is, $\mathcal{P}$ consists of R and the $k-1$ lines not concurrent with R. An automorphism θ of $\mathcal{N}$ is called a *transvection with axis* R if either $\theta = \mathbf{1}$ or if $\mathcal{P}$ is the set of all fixed lines of θ and R is the set of all fixed points of θ. The net $\mathcal{N}$ is a *$\mathcal{P}$-net* [128] if for any two non-parallel lines $M, N \in \overline{B} \setminus \mathcal{P}$ there is some transvection with axis belonging to $\mathcal{P}$ and mapping M onto N.
In particular, let $\mathcal{N} = (\overline{P}, \overline{B}, \overline{I})$ be an affine plane of order k and let $\mathcal{D}$ be the corresponding projective plane. Then $\mathcal{N}$ is a $\mathcal{P}$-net if and only if the point z of $\mathcal{D}$ defined by $\mathcal{P}$ is a translation point of $\mathcal{D}$.

If $\mathcal{N}$ is the dual of H_q^n then it is easy to check that $\mathcal{N}$ is a $\mathcal{P}$-net for any parallel class $\mathcal{P}$.

We now state the following results without proofs.

Theorem 4.4.2 (J. A. Thas [128]). *Let $(\mathcal{S}^{(p)}, G)$ be a TGQ of order (s,t), $s \neq 1 \neq t$. Then for any line L incident with p, the dual net $\mathcal{N}_L^*$ defined by L is the dual of a $\mathcal{P}$-net $\mathcal{N}_L$ with $\mathcal{P}$ the parallel class of $\mathcal{N}_L$ defined by the point p.* ∎

Theorem 4.4.3 (J. A. Thas [128]). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s,t), $s \neq 1 \neq t$, with coregular point p. If for at least one line L incident with p the dual net $\mathcal{N}_L^*$ is the dual of a $\mathcal{P}$-net $\mathcal{N}_L$ with $\mathcal{P}$ the parallel class of $\mathcal{N}_L$ defined by p, then $\mathcal{S}$ is a TGQ with base-point p.* ∎

Corollary 4.4.4 (J. A. Thas [128]). *Let $\mathcal{S}$ be a GQ of order s, $s \neq 1$, with coregular point p. If for at least one line L incident with p the projective plane π_L is Desarguesian, then $\mathcal{S}$ is a TGQ with base-point p. If in particular s is odd, then $\mathcal{S}$ is isomorphic to the classical GQ $\mathcal{Q}(4,s)$.* ∎

Corollary 4.4.5 (J. A. Thas [128]). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (q^2, q), $q \neq 1$, with regular point x for which the dual net $\mathcal{N}_x^*$ defined by x satisfies the Axiom of Veblen. If x is incident with a coregular line L, then $\mathcal{S}$ is a TGQ with base-line L.* ∎

Chapter 5

Symmetry-Class I: Generalized Quadrangles without Axes of Symmetry

The Symmetry-Class I. *We say that a GQ is contained in Symmetry-Class* **I** *if it contains no axes of symmetry.*

We note that a GQ of order (s, t), $s \neq 1 \neq t$, with $t < s$ cannot have axes of symmetry since an axis of symmetry is a regular line.

5.1 The Classical and Dual Classical Examples

The following classical or dual classical GQ's contain no axes of symmetry: $H(3, q^2)$, $H(4, q^2)^D$, $W(q)$ with q odd, and $H(4, q^2)$.

5.2 The GQ's $\mathcal{P}(\mathcal{S}, x)$ of S. E. Payne

Let $\mathcal{S}$ be a GQ of order s, $s > 1$, with regular point x. Construct $\mathcal{S}' = \mathcal{P}(\mathcal{S}, x)$ as in Chapter 3, and suppose θ is a nontrivial symmetry about some line in $\mathcal{S}'$. Then by Theorem 1.2.1, we have that

$$(s+1)(s-1)s \equiv 0 \mod 2s,$$

and hence s must be odd[1]. We have a look at the known GQ's $\mathcal{P}(\mathcal{S}, x)$ of order $(s-1, s+1)$ (i.e. those coming from known $\mathcal{S}$), s odd.

[1]Note that this works with each thick GQ of order $(s-1, s+1)$.

The only known GQ's of order s (without restriction on the parity) with a regular point are:

(i) the dual of a $T_2(\mathcal{O})$ of Tits of order s ($\mathcal{O}$ a known oval of $\mathbf{PG}(2,s)$);

(ii) a $T_2(\mathcal{O})$ of Tits of order s with s even.

If we are in Case (i) and s is odd, then $T_2(\mathcal{O})^D \cong W(q)$, $s = q$ an odd prime power. The following is due to S. E. Payne and J. A. Thas:

Theorem 5.2.1 (FGQ, 3.3.5). *Suppose L and M are distinct non-concurrent lines of the GQ $\mathcal{P}(W(q),x)$, where $q > 3$, q odd. Then $\{L,M\}$ is a regular pair if and only if one of the following holds:*

(i) *L and M are concurrent lines in $W(q)$;*

(ii) *L and M are hyperbolic lines of $W(q)$ which contain x.* ■

Hence, this result yields the fact that the GQ's $\mathcal{P}(W(q),x)$, $q > 3$ and q odd, x any regular point of $W(q)$, have no regular lines. Hence no axes of symmetry.

Theorem 5.2.2. *The GQ $\mathcal{P}(\mathcal{S},x)$ is an element of the Symmetry-Class* **I** *for all the known GQ's $\mathcal{S}$ of order s with regular point x if $s > 3$ and s is odd. If s is even and $s > 2$, then $\mathcal{P}(\mathcal{S},x)$ is always a member of* **I**. *If $s = 3$, then any $\mathcal{P}(\mathcal{S},x)$ is isomorphic to $\mathcal{Q}(5,2)$, and then every line is an axis of symmetry (and so $\mathcal{P}(\mathcal{S},x)$ is not a member of* **I**). *If $s = 2$, then $\mathcal{S} \cong \mathcal{Q}(4,2)$, and $\mathcal{P}(\mathcal{S},x)$ is just a plain dual grid with parameters $4,4$.* ■

The GQ's $\mathcal{P}(\mathcal{S},x)^D$ are also members of **I** since there are more points on a line than lines through a point (a $\mathcal{P}(\mathcal{S},x)^D$ cannot have regular lines). In fact, suppose $\mathcal{S}$ is any thick GQ of order $(s+1,s-1)$ which admits a nontrivial symmetry about some line. Then

$$(s+1)(s-1)(s+2) \equiv 0 \mod 2s,$$

forcing $s = 2$.

5.3 Regularity (for Lines) in the GQ's $\mathcal{P}(\mathcal{S},x)$

In this section, we push our results a little further in a more general context, and we have a brief look at regularity (for lines) in the GQ's $\mathcal{P}(\mathcal{S},x)$. As will be shown, this will be very useful in the context of the classification. We will come to a rather remarkable observation which links several combinatorial problems to each other.

The following observation is an easy generalization of Theorem 5.2.1, but the proof is (necessarily) completely different.

Observation 5.3.1. *Suppose L and M are two distinct non-concurrent lines of the GQ $\mathcal{P}(\mathcal{S},x)$, where $\mathcal{S}$ is of order $s > 3$. Suppose that one of the following holds:*

(i) *L and M are distinct concurrent lines in $\mathcal{S}$;*

(ii) *L and M are distinct hyperbolic lines of $\mathcal{S}$ which contain x.*

Then $\{L, M\}$ is a regular pair of lines.

Proof. If we are in Case (i), then it is an easy exercise to show that $\{L, M\}$ is a regular pair of lines (the lines of Type (a) of $\mathcal{P}(\mathcal{S}, x)$ through $L \cap M$ form $\{L, M\}^{\perp\perp}$). Now suppose we are in Case (ii), and put $L = P_L$ and $M = P_M$, where P_L and P_M are the corresponding point sets in $\mathcal{S}$. By Theorem 4.1.1, we know that $P_L^{\perp} \cap P_M^{\perp}$ is a point r in $x^{\perp} \setminus \{x\}$, and clearly every line in $\mathcal{S}$ through r and different from rx forms a line of $\{L, M\}^{\perp}$ in $\mathcal{P}(\mathcal{S}, x)$. By (i), we conclude that $\{L, M\}$ is regular. ∎

Now suppose $\{L, M\}$ is a regular pair of non-concurrent lines which is not of one of the Types (i) and (ii) of Observation 5.3.1. By the previous observations, we know that if H and H' are distinct hyperbolic lines through x in $\mathcal{S}$, then

(a) they form a regular pair of lines in $\mathcal{P}(\mathcal{S}, x)$;

(b) $\{H, H'\}^{\perp}$ (where '$\perp$' is taken in $\mathcal{P}(\mathcal{S}, x)$) consists only of lines of Type (a);

(c) $\{H, H'\}^{\perp\perp}$ only consists of lines of Type (b).

Hence we can suppose that L is of Type (a), that no line of Type (a) of $\{L, M\}^{\perp\perp} \setminus \{L\}$ (in $\mathcal{P}(\mathcal{S}, x)$) intersects L in $\mathcal{S}$, and that $\{L, M\}^{\perp\perp}$ contains at most one line of Type (b). Thus, $\{L, M\}^{\perp\perp}$ has at least $s - 1$ lines of Type (a) which are mutually non-concurrent in $\mathcal{S}$. The same can be concluded for $\{L, M\}^{\perp}$. Hence the lines of Type (a) in $\{L, M\}^{\perp} \cup \{L, M\}^{\perp\perp}$ form ('at least') a grid with parameters $s-1, s-1$ in $\mathcal{S}$. It is also clear that $\{L, M\}^{\perp}$ and $\{L, M\}^{\perp\perp}$ cannot contain a line of Type (b) at the same time since such lines are disjoint in $\mathcal{P}(\mathcal{S}, x)$. Now suppose that the 'missing lines' in $\{L, M\}^{\perp}$ and $\{L, M\}^{\perp\perp}$ are both of Type (a). From Chapter 4 of K. Thas [149] we know that a grid with parameters k, s, with $k > 2$, in a thick GQ of order (s, t), is always contained in a grid with parameters $k, s + 1$. Hence, $\{L, M\}^{\perp} \cup \{L, M\}^{\perp\perp}$ is contained in a grid with parameters $s + 1, s + 1$ of $\mathcal{S}$, that is, $\mathcal{S}$ contains a regular pair of lines.
We know that at least one of the missing lines in $\{L, M\}^{\perp}$ and $\{L, M\}^{\perp\perp}$ is of Type (a). Suppose now the other is of Type (b). Again from [149] we conclude that $\{L, M\}^{\perp} \cup \{L, M\}^{\perp\perp}$ is contained in a grid with parameters $s - 1, s + 1$ of $\mathcal{S}$, and one observes that this grid is complete. So by Chapter 4 of K. Thas [149], we have that $\mathcal{S}$ contains a dual complete $(s^2 - 1)$-arc (and a spread).

We now have the following.

Observation 5.3.2. *Consider the GQ $\mathcal{P}(\mathcal{S}, x)$ which arises from the GQ $\mathcal{S}$ of order s, $s > 1$, with regular point x. Suppose L is a regular line in $\mathcal{P}(\mathcal{S}, x)$. Then we have that $\mathcal{S}$ contains a dual complete $(s^2 - 1)$-arc. Hence if $\mathcal{S}$ is a known GQ of order $s > 1$, then we have one of the following possibilities:*

(i) $\mathcal{S} \cong \mathcal{Q}(4,2)$ *and up to isomorphism there is a unique example;*

(ii) $\mathcal{S} \cong W(s)$ *with* s *odd.*

Proof. First suppose that L is of Type (a) and that L is a regular line in $\mathcal{S}$. Let $M \not\sim L$ be a line of Type (b). Then by preceding considerations one easily constructs a dual complete (s^2-1)-arc from $\{L,M\}^{\perp\perp}$ (considered in $\mathcal{P}(\mathcal{S},x)$).
Suppose that L is of Type (a) and that L is not a regular line in $\mathcal{S}$. Then by preceding considerations, we have that $\{L,M\}^{\perp} \cup \{L,M\}^{\perp\perp}$ is contained in a complete grid Γ of $\mathcal{S}$ with parameters $s-1, s+1$, if M is a line of Type (a) not meeting L in $\mathcal{S}$. Now consider the lines of $\mathcal{S}$ which do not intersect lines of this grid. Then together with the $s-1$ mutually non-concurrent lines of Γ of one type, they form a dual complete (s^2-1)-arc $\mathcal{U}$ of $\mathcal{S}$, and Theorem 1.9.5 applies.
The case where L is of Type (b) follows in the same way. ∎

Since in a $\mathcal{P}(\mathcal{S},x)$ of order $(s-1,s+1)$, axes of symmetry only can exist for s odd, Observation 5.3.2 yields that existence of axes of symmetry in $\mathcal{P}(\mathcal{S},x)$ implies existence of 'nontrivial' complete (s^2-1)-arcs for $s>3$.

Remark 5.3.3. (i) In [25], B. De Bruyn and S. E. Payne show that for a GQ $\mathcal{S}$ of order $s>1$ with regular point x which is also coregular (and then by Theorem 1.4.3(ii) s is even), $\mathcal{P}(\mathcal{S},x)$ has the property that any two distinct concurrent lines are contained in a unique grid of $\mathcal{P}(\mathcal{S},x)$ with parameters s,s. Hence in that case, there are no regular lines.

(ii) It is not known when a collineation θ of $\mathcal{P}(\mathcal{S},x)$ induces a collineation θ' of $\mathcal{S}$ (which fixes x). Of course, a necessary and sufficient condition is that θ preserves the types of the lines (and clearly, any collineation θ' of $\mathcal{S}$ which fixes x induces an automorphism of $\mathcal{P}(\mathcal{S},x)$). It should not be too hard to solve this question. A recent reference on this problem is S. E. Payne and M. Miller [87], see also [11, 28, 29]. (The references [11, 28, 29], for instance, easily yield the information that the collineation groups of the GQ's which arise from hyperovals in $\mathbf{PG}(2,q)$, q even, are completely determined by the stabilizer in $\mathbf{P\Gamma L}(4,q)$ of the associated hyperoval.)

(iii) Suppose L is a line of Type (a) of $\mathcal{P}(\mathcal{S},x)$; if a nontrivial symmetry about L in $\mathcal{P}(\mathcal{S},x)$ would induce an automorphism of $\mathcal{S}$, then this would be a nontrivial symmetry about L in $\mathcal{S}$ which fixes x, clearly a contradiction. Next, suppose that L is a line of Type (b) in $\mathcal{P}(\mathcal{S},x)$ which is an axis of symmetry in $\mathcal{P}(\mathcal{S},x)$, with corresponding (full) group of symmetries G_L that extends to a group of automorphisms of $\mathcal{S}$. Then clearly G_L induces a group of collineations G'_L of $\mathcal{S}$ of size $s-1$ which fixes $\{x,p\}^{\perp} \cup \{x,p\}^{\perp\perp}$ pointwise, where $\{x,p\}^{\perp\perp}$ corresponds to L in $\mathcal{S}$. Moreover, by Theorem 1.3.3, G'_L acts regularly on the $s-1$ points of any line N of $\mathcal{S}$ incident with some point of $\{x,p\}^{\perp} \cup \{x,p\}^{\perp\perp}$, and which are not contained in $\{x,p\}^{\perp} \cup \{x,p\}^{\perp\perp}$.

Let M be an arbitrary line of $\mathcal{S}$ which is not incident with some point of $\{x,p\}^\perp \cup \{x,p\}^{\perp\perp}$. Then $M^{G'_L}$ together with all the lines of $\mathcal{S}$ which are incident with a point of $\{x,p\}^{\perp\perp}$ and which hit M, form a complete grid $\mathcal{G}$ with parameters $s-1, s+1$. Now consider the lines of $\mathcal{S}$ which do not intersect a line of $(M^{G'_L})^\perp$. Then together with $M^{G'_L}$ they form a dual complete (s^2-1)-arc $\mathcal{U}$ of $\mathcal{S}$.

5.4 The Other Known (Non-Classical) GQ's of Order (s,t) with $t < s$

For any flock $\mathcal{F}$ of the quadratic cone of $\mathbf{PG}(3,q)$, $\mathcal{S}(\mathcal{F})$ is an element of $\mathbf{I}$ since $\mathcal{S}(\mathcal{F})$ is of order (q^2,q).
Also, for any ovoid $\mathcal{O}$ of $\mathbf{PG}(3,q)$, the GQ $T_3(\mathcal{O})^D$ is contained in $\mathbf{I}$.

Remark 5.4.1. The following GQ's (or, more precisely, GQ types) all have *at least one* axis of symmetry (see further for more details):

(i) a $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$, where $\mathcal{O}$ is an oval, respectively ovoid, of $\mathbf{PG}(2,q)$, respectively $\mathbf{PG}(3,q)$ (note that $T_3(\mathcal{O})^* \cong T_3(\mathcal{O})$);

(ii) the dual $T_2(\mathcal{O})^D$ of a $T_2(\mathcal{O})$ of Tits of order s, s even;

(iii) any GQ $\mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ a flock of the quadratic cone;

(iv) the translation duals $(\mathcal{S}(\mathcal{F})^D)^*$ of the point-line duals of semifield flock GQ's $\mathcal{S}(\mathcal{F})$, where in the known examples $\mathcal{F}$ is a Ganley flock, a Penttila-Williams flock or a Kantor flock (note that in the last case $(\mathcal{S}(\mathcal{F})^D)^* \cong \mathcal{S}(\mathcal{F})^D$).

Chapter 6

Symmetry-Class II: Concurrent Axes of Symmetry in Generalized Quadrangles

The Symmetry-Class II. *We say that a GQ is contained in Symmetry-Class* **II** *if all its axes of symmetry are incident with the same point p and if there is at least one axis of symmetry.*

This class is a very large class of GQ's, which in fact contains most of the known examples of GQ's (up to duality). We first observe that if x is the point of an arbitrary element $\mathcal{S}$ of **II** which is incident with the $k+1$ axes of symmetry $L_0, L_1, \ldots, L_k$ of $\mathcal{S}$, then any automorphism of $\mathcal{S}$ must fix x if $k > 0$. The set $\{L_0, L_1, \ldots, L_k\}$ is always fixed by any automorphism of $\mathcal{S}$.

In FGQ it was proved that a (thick) GQ $\mathcal{S}$ is a TGQ $(\mathcal{S}^{(p)}, G)$ with translation point p if and only if every line through p is an axis of symmetry, that is, if p is a translation point, and then G is precisely the group generated by all symmetries about the lines incident with p, cf. Theorem 2.1.4. In our Master Thesis [141] we noted that that same proof was also valid for all lines through p minus one. This observation is one of the main motivations for the present chapter and the essential question in the study of Symmetry-Class **II**:

What is — in general — the minimal number of distinct axes of symmetry through a point p of a GQ $\mathcal{S}$ forcing $\mathcal{S}^{(p)}$ to be a TGQ?

Prior to [141], there were only such results known for (thick) GQ's of order s, and in that case, three axes of symmetry appears to be sufficient. In an appendix, we will give a short new geometrical proof of this theorem without using coordinatization methods for GQ's, but using results of X. Chen and D. Frohardt[1]. For thick GQ's

[1]The known proof of this theorem is contained in Chapter 11 (Theorem 11.3.5 of that chapter)

of order (s,t) with $s \neq t$, the problem is a lot harder; we will show that $t-s+3$ axes of symmetry are sufficient in the general case.
In order to study the generalized quadrangles which have some distinct axes of symmetry through some point p, we will introduce "Property (T)". TGQ's which satisfy Property (T) for some ordered flag always have order (s,s^2) for some s, and every TGQ of order (s,t) which has a (proper) subGQ of order s through the translation point satisfies Property (T) for some ordered flag(s) containing the translation point. Property (T) is closely related to Property (G), and seems to be more general in the case of translation generalized quadrangles. Moreover, every known translation generalized quadrangle (of suitable order) or its translation dual has Property (T). Suppose that the GQ $\mathcal{S}$ satisfies Property (T) for the ordered flag (L,p) w.r.t. the distinct lines L_1, L_2, L_3, all incident with p. Moreover, suppose that L, L_1, L_2, L_3 are axes of symmetry. Then we will show that $\mathcal{S}^{(p)}$ is a TGQ and that the translation group G is generated by the symmetries about L, L_1, L_2, L_3.
We will also study the following related problem:

Given a general TGQ $\mathcal{S}^{(p)}$, what is the minimal number of lines through p such that the translation group is generated by the symmetries about these lines?

We will show that there is a connection between the minimal number of lines through a translation point of a TGQ such that the translation group is generated by the symmetries about these lines, and the kernel of the TGQ; in particular, if $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s,t), $1 \neq s \neq t \neq 1$, with $(s,t) = (q^{na}, q^{n(a+1)})$, where a is odd and where $\mathbf{GF}(q)$ is the kernel of the TGQ, and if $k+3$ is the minimal number of distinct lines through p such that G is generated by the symmetries about these lines, then we will show that $k \leq n$. We will also introduce "Property (T$'$)". A goal of this chapter is to state elementary combinatorial and group theoretical conditions for a GQ $\mathcal{S}$ such that $\mathcal{S}$ arises from a flock. We will show that a combination of Property (T$'$) and Property (T) leads to Property (G) for TGQ's $\mathcal{S}$, and hence that it is possible to prove that such a TGQ $\mathcal{S}$ is related to a flock. A classification result is eventually obtained.
Many other results will be proved, including a new (easy to obtain) divisibility condition for GQ's which have a point incident with at least three axes of symmetry, see Section 6.4.

Remark. The results of this chapter are also motivated by the following problem, which we see as the fourth in the series of problems posed in Section 2.1:

(4) *Suppose G is a group which is generated by elations about the same point x of a thick GQ $\mathcal{S}$. When is G a group of elations?*

We will provide several 'structure theorems' where sufficient conditions for such groups will be stated in order to be elation groups.

The results (with proofs) of "Part (b): Symmetry-Class **II.2**" of this chapter up to Section 6.7, and the addendum of the chapter, are based on K. Thas [148].

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of FGQ, is rather long and technical and uses coordinatization.

Part (a): Symmetry-Class II.1

The Symmetry-Class II.1. *Each element of* **II.1** *is defined to have exactly one axis of symmetry.*

6.1 Some Observations

Suppose $\mathcal{S}$ is a (thick) element of **II.1** of order (s,t) with axis of symmetry L. Then by Theorem 2.3.16, p is not an elation point if pIL and if s is even. By Theorem 1.2.1 we also have that

$$st(s+1) \equiv 0 \mod s+t.$$

Each flock GQ $\mathcal{S}(\mathcal{F})$ contains a center of symmetry by Theorem 2.7.5 (namely the point (∞)), hence for every flock $\mathcal{F}$ of the quadratic cone of $\mathbf{PG}(3,q)$, the GQ $\mathcal{S}(\mathcal{F})^D$ of order (q,q^2) contains at least one axis of symmetry. If there is a center of symmetry of $\mathcal{S}(\mathcal{F}) = (P,B,I)$ in $P \setminus (\infty)^\perp$, then every point of $\mathcal{S}(\mathcal{F})$ is a center of symmetry (the point (∞) is an elation point), and $\mathcal{S}(\mathcal{F})$ (and then also $\mathcal{S}(\mathcal{F})^D$) is classical, i.e. $\mathcal{F}$ is linear. If $\mathcal{F}$ is a Kantor flock, a Penttila-Williams flock or a Ganley flock, then by Theorem 10.3.1 (see Chapter 10), $(\mathcal{S}(\mathcal{F})^D)^*$ contains a line $[\infty]$ which is a line of translation points, i.e. each line of $[\infty]^\perp$ is an axis of symmetry. If $\mathcal{F}$ is a Kantor flock, then $\mathcal{S}(\mathcal{F})^D \cong (\mathcal{S}(\mathcal{F})^D)^*$ by Theorem 3.4.1. By Theorem 10.10.1, $\mathcal{S}(\mathcal{F})^D$ contains exactly one translation point if $\mathcal{F}$ is a non-classical (i.e. nonlinear) Ganley flock (and then $s > 9$), or a Penttila-Williams flock.
More generally:

Observation 6.1.1. *If $\mathcal{S}(\mathcal{F})^D$ is a TGQ (for some point), $\mathcal{F}$ not a Kantor flock, then $\mathcal{S}(\mathcal{F})^D$ always has precisely one translation point.* ∎

Observation 6.1.2. *If $\mathcal{F}$ is a known flock of the quadratic cone of $\mathbf{PG}(3,q)$ and $\mathcal{F}$ is not one of the previous flocks (or, more generally, if $\mathcal{S}(\mathcal{F})^D$ is not a TGQ for some point), then $\mathcal{S}(\mathcal{F})^D$ is an element of* **II.1**. ∎

The following theorem considers the possible subconfigurations of axes of symmetry for the point-line dual of the $T_2(\mathcal{O})$ of Tits.

Theorem 6.1.3. *Suppose $\mathcal{S}$ is isomorphic to a $T_2(\mathcal{O})^D$ of order $q > 1$, $\mathcal{O}$ an oval of $\mathbf{PG}(2,q)$. If q is odd, then no line of $\mathcal{S}$ is an axis of symmetry. Suppose q is even. Then $\mathcal{S} = \mathcal{S}^{(x)}$ is a TGQ for some point x if and only if $\mathcal{O}$ is a translation oval of $\mathbf{PG}(2,q)$, and then $\mathcal{S}^{(x)} \cong T_2(\mathcal{O})$. In particular, if $\mathcal{O}$ is a conic, then every line of $\mathcal{S}$ is an axis of symmetry. If $\mathcal{O}$ is not a translation oval of $\mathbf{PG}(2,q)$, then $\mathcal{S} \cong T_2(\mathcal{O})^D$ has exactly one axis of symmetry.*

Proof. If q is odd, then by the celebrated result of B. Segre (Theorem 1.7.1), $\mathcal{O}$ is a conic and then $T_2(\mathcal{O}) \cong \mathcal{Q}(4,q)$, and hence by Theorem 1.5.1, $\mathcal{S} \cong W(q)$.

Since no line of $W(q)$ is regular if q is odd, $\mathcal{S}$ cannot contain axes of symmetry. Suppose q is even, and consider $T_2(\mathcal{O})$. Then (∞) is an elation point (since it is a translation point), and as a coregular point, (∞) is regular since q is even (by Theorem 1.4.4(iv)). By Corollary 4.2.10, (∞) is a center of symmetry. Suppose $T_2(\mathcal{O})$ contains another center of symmetry u. If $u \not\sim (\infty)$, then $\mathcal{S} \cong W(q)$ by Theorem 1.5.2, as each point of $\mathcal{S}$ is a center of symmetry, and hence regular. If $u \sim (\infty)$, clearly $u(\infty)$ is a line each point of which is a center of symmetry, and hence $\mathcal{S}$ is a TGQ with base-line $u(\infty)$. Also, if $T_2(\mathcal{O})$ is non-classical, there are no other centers of symmetry v in this case; otherwise we would have the following possibilities.

(a) We have that $v \not\sim (\infty)$. Then each point is a center of symmetry and hence regular — contradiction by Theorem 1.5.2.

(b) We have that $v \sim (\infty)$. Then each point of $(\infty)^\perp$ is a center of symmetry and hence regular. It follows that each point is regular (cf. Theorem 1.4.2(iv)), which produces the same contradiction as before.

Now suppose that $\mathcal{S} = \mathcal{S}^{(x)}$ is a TGQ for the point x. Suppose $[\infty]$ corresponds to the translation point (∞) of $T_2(\mathcal{O})$ under some duality δ.[2] First suppose $x \not{I} [\infty]$. As there is an elation about $[\infty]$ which maps some axis of symmetry through x onto a non-concurrent axis of symmetry, each line is regular by the dual of Theorem 1.4.2(iv), and then $\mathcal{S} \cong \mathcal{Q}(4,q)$ by Theorem 1.5.2. So suppose $xI[\infty]$. Apply δ to obtain, in the obvious notation, $[X]I(\infty)$, where $[X]$ is a translation line. We now work in $\mathbf{PG}(3,q)$, where $\mathcal{O}$ lies in some fixed $\mathbf{PG}(2,q)$. Recall from Theorem 2.4.4 that an automorphism of $T_2(\mathcal{O})$ which fixes (∞) is induced by an automorphism of $\mathbf{PG}(3,q)$ which fixes $\mathcal{O}$ (and $\mathbf{PG}(2,q)$). Every point on $[X]$ is a center of symmetry and hence, if $p_{[X]}$ is the point of $\mathcal{O}$ which corresponds to $[X]$, then for each plane π of $\mathbf{PG}(3,q)$ through $p_{[X]}$ which intersects $\mathcal{O}$ only in $p_{[X]}$, there is a group H_π of collineations of $\mathbf{PG}(3,q)$ which stabilizes $\mathcal{O}$, which fixes π pointwise, and which acts regularly on $\mathcal{O} \setminus p_{[X]}$. Hence $\mathcal{O}$ is a translation oval w.r.t. $p_{[X]}$, and so $\mathcal{S}^{(x)} = T_2(\mathcal{O})^D \cong T_2(\mathcal{O})$ by Theorem 1.8.3. Finally, if $\mathcal{O}$ is a conic, then $T_2(\mathcal{O}) \cong \mathcal{Q}(4,q)$, and $\mathcal{Q}(4,q)$ is isomorphic to $W(q)$ if and only if q is even, see Theorem 1.5.1. ■

There is a nice corollary which characterizes translation ovals $\mathcal{O}$ of $\mathbf{PG}(2,q)$, q even, in terms of the $T_2(\mathcal{O})$'s which arise from it.

Corollary 6.1.4. *Suppose $\mathcal{O} \subseteq \mathbf{PG}(2,q)$ is an oval of $\mathbf{PG}(2,q)$, where q is even. Then $\mathcal{O}$ is a translation oval if and only if $T_2(\mathcal{O})^D$ is a TGQ for some translation point x, in which case $T_2(\mathcal{O})^D \cong T_2(\mathcal{O})$.* ■

Theorem 6.1.3 justifies the following conjecture:

[2]The only reason why we mention this duality is formal; in the classical case the choice of $[\infty]$ is not well-defined (it is arbitrary).

Conjecture. *A TGQ of order s, $s > 1$, for which the point-line dual is also a TGQ (w.r.t. some translation point), is isomorphic to a $T_2(\mathcal{O})$ of Tits, where $\mathcal{O}$ is a translation oval of $\mathbf{PG}(2,s)$, s a power of 2.*

An affirmation of the latter conjecture would lead to the solution of the following weaker conjecture:

Conjecture. *A TGQ of order s, $s > 1$, is self-dual if and only if it is isomorphic to a $T_2(\mathcal{O})$ of Tits, where $\mathcal{O}$ is a translation oval of $\mathbf{PG}(2,s)$, s a power of 2.*

Part (b): Symmetry-Class II.2

The Symmetry-Class II.2. *Each element of* **II.2** *has exactly $k+1$ concurrent axes of symmetry (through some point p), with $1 \leq k \leq t-1$.*

6.2 Property (T)

We start with defining *Property (T)*.
Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and let p be a point of the GQ.

Property (T). An ordered flag (L,p) satisfies *Property (T) with respect to L_1, L_2, L_3*, where L_1, L_2, L_3 are three distinct lines incident with p and distinct from L, if the following condition holds:
if (i,j,k) is a permutation of $(1,2,3)$, if $M \sim L$ and $M \not\mathrm{I} p$, and if $N \sim L_i$ and $N \not\mathrm{I} p$ with $M \not\sim N$, then the triads $\{M,N,L_j\}$ and $\{M,N,L_k\}$ are not both centric.
If the ordered flag (L,p) satisfies Property (T) with respect to L_1, L_2, L_3, then we also say that $\mathcal{S}$ *satisfies Property (T) for (or at) the ordered flag (L,p) w.r.t. L_1, L_2, L_3.*

The following theorem will turn out to be very useful throughout.

Theorem 6.2.1. *Suppose $\mathcal{S} = (P,B,I)$ is a thick GQ of order (s,t), and let p be a point of $\mathcal{S}$ incident with three distinct axes of symmetry. If G is the group generated by the symmetries about these lines, then G is a group of order s^3 of elations with center p.*

Proof. Suppose that L_1, L_2, L_3 are three distinct axes of symmetry incident with p, and let G_i be the full group of symmetries about L_i, $i = 1,2,3$. With α_1, α_2 and β contained in respectively G_1, G_2 and G_3 (and none of these collineations trivial), suppose that the following holds:

$$\alpha_1\alpha_2 = \beta.$$

If q is not collinear with p, then $(q, q^\beta, q^{\alpha_1})$ are the points of a triangle, a contradiction. This observation shows us that $|G| = s^3$ (cf. Theorem 1.2.2), and that each element of G is an elation (because no two elements of G have the same action on a point of $P \setminus p^\perp$), and so G is a group of elations about p. ∎

In the following, Theorem 1.2.2 will often be used without further notice.
We now obtain

Theorem 6.2.2. *Suppose $\mathcal{S} = (P, B, I)$ is a thick GQ of order (s, t), and let $p \in P$ be a point incident with four distinct axes of symmetry L_1, L_2, L_3 and L_4. Moreover, suppose that Property (T) holds for the ordered flag (L_4, p) w.r.t. the lines L_1, L_2, L_3. Then $t = s^2$, every line through p is an axis of symmetry, and $(\mathcal{S}^{(p)}, G)$ is a TGQ, where G is the group generated by all symmetries about L_1, L_2, L_3, L_4.*

Proof. Suppose that G_i is the full group of symmetries about the line L_i, $i \in \{1, 2, 3, 4\}$, and consider the group G generated by all symmetries about the lines L_1, L_2, L_3 and L_4. We define G' as the group generated by the symmetries about the lines L_j with $j = 1, 2, 3$. A general element of G' always can be written in the form $\phi_1\phi_2\phi_3$, with ϕ_k a symmetry about the line L_k, $k = 1, 2, 3$, see Theorem 1.2.2. Also, because of Theorem 6.2.1, G' is an elation group of elations with center p and of size s^3.
Suppose, for a nontrivial symmetry $g \in G_4$ and an elation $\alpha = \alpha_1\alpha_2\alpha_3$ of G', with $\alpha_i \in G_i$, that g and α have the same action on a point q of $P \setminus p^\perp$. Then $q^g = q^\alpha =: q'$, and thus we have that $q' \in [q, L_4]$ (note that $q' \neq q$). It is clear that none of the symmetries α_i is trivial ($i = 1, 2, 3$), otherwise we would have a product of at most three symmetries about distinct axes of symmetry with a common intersection point which acts trivially on a point of $P \setminus p^\perp$, a contradiction by Theorem 6.2.1. If we now consider the triads of lines $\{[q, L_4], [q^{\alpha_1}, L_2], L_3\}$ and $\{[q, L_4], [q^{\alpha_1}, L_2], L_1\}$, then the assumption we just made implies that they are both centric, in contradiction with Property (T). Thus we have proved that an element of G' and an element of G_4 can never have the same action on a point of $P \setminus p^\perp$. Therefore, we have that $|G| = s^4$ and that every element of G is an elation with center p. Since $|P \setminus p^\perp| = s^2 t$ and since $t \leq s^2$, it follows that $t = s^2$. Also, G acts regularly on the points of $P \setminus p^\perp$, and hence $(\mathcal{S}^{(p)}, G)$ is an EGQ with elation point p. Since the lines L_i are regular, the proof is complete by Theorem 2.3.15. ■

Recall, for the following theorem, that if $\mathcal{S}$ is a GQ of order (s, s^2), $s > 1$, and if $\mathcal{S}'$ is a subquadrangle of $\mathcal{S}$ of order s, then every line of $\mathcal{S}$ is either contained in $\mathcal{S}'$, or intersects $\mathcal{S}'$ in exactly one point.

Theorem 6.2.3. *Suppose $\mathcal{S}$ is a thick GQ of order (s, t), and let p be a point incident with four distinct axes of symmetry L_1, L_2, L_3, L_4, three of which are contained in a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order (s, t'), but not the fourth, say L_4. Then (L_4, p) satisfies Property (T) w.r.t. L_1, L_2, L_3. Thus $t = s^2$ and $(\mathcal{S}^{(p)}, G)$ is a TGQ, with G the group generated by all symmetries about these four lines. Moreover, $\mathcal{S}'$ also is a TGQ (with translation point p).*

Proof. It is clear that an axis of symmetry of a GQ of order (s, t) also is an axis of symmetry of a proper subGQ of order (s, t') which contains this line (cf. Lemma 1.3.5), $1 < t' < t$ and $s \neq 1$. An axis of symmetry of a GQ is a regular line, and since $t' \neq 1$, Theorem 1.3.1 implies that $s \leq t'$ so that $t' = s$. Hence $t = s^2$. By Theorem 2.3.3, $\mathcal{S}'$ is a TGQ.

Let L_1, L_2, L_3 be three distinct axes of symmetry of $\mathcal{S}$ through the point p which are contained in $\mathcal{S}'$, and let L_4 be an axis of symmetry of $\mathcal{S}$ through p which is not contained in $\mathcal{S}'$. Suppose $L \sim L_4 \neq L$ is arbitrary, $L \not{I} p$, and let $L' \not\sim L$ be an arbitrary line of $L_i^{\perp} \setminus \{L_i\}$ not through p for a fixed i in $\{1,2,3\}$. Suppose that the triads of lines $\{L, L', L_j\}$ and $\{L, L', L_k\}$ are both centric, with $\{j,k\} = \{1,2,3\} \setminus \{i\}$. First suppose that $L' \in \mathcal{S}'$. If M is a center of $\{L, L', L_j\}$, then $|M \cap \mathcal{S}'| \geq 2$, and hence M is a line of $\mathcal{S}'$. It follows that $|L \cap \mathcal{S}'| \geq 2$, and hence L also is a line of $\mathcal{S}'$. This immediately leads to the fact that L_4 is a line of $\mathcal{S}'$, a contradiction. (The same holds for $\{L, L', L_k\}$.)
Next, suppose that $L' \notin \mathcal{S}'$, that $M \in \{L, L', L_j\}^{\perp}$ and $N \in \{L, L', L_k\}^{\perp}$. The line L intersects $\mathcal{S}'$ in one point q. Consider a symmetry θ about L_4 which maps the point $q' = L \cap M$ onto q. Then the line M^{θ} is contained in $\mathcal{S}'$, and hence also $(L')^{\theta}$. By the first part of this proof, we now obtain a contradiction. Hence Property (T) is satisfied for the ordered flag (L_4, p) w.r.t. the lines L_1, L_2, L_3, and the theorem follows from Theorem 6.2.2. ∎

6.3 Property (T′)

Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and let p be a point of the GQ.

Property (T′). An ordered flag (L,p) satisfies *Property (T′) with respect to the lines* L_1, L_2, L_3, where L_1, L_2, L_3 are different lines incident with p and distinct from L, if the following condition holds:
if $M \sim L$ and $M \not{I} p$, and if q and q' are distinct arbitrary points on M which are not incident with L, then there is a permutation (i,j,k) of $(1,2,3)$ such that there are lines M_i, M_j, M_k, with $M_r \sim L_r$ and $r \in \{i,j,k\}$, for which $M_j \in \{M_i, M_k, L_j\}^{\perp}$, and such that qIM_i and $q'IM_k$.
If the ordered flag (L,p) satisfies Property (T′) w.r.t. the lines L_1, L_2, L_3, then we also say that $\mathcal{S}$ satisfies *Property (T′) for (or at) the ordered flag (L,p) w.r.t. L_1, L_2, L_3.*

We now have

Theorem 6.3.1. *Suppose that $\mathcal{S} = (P,B,I)$ is a thick GQ of order (s,t), and let p be a point of P which is incident with four distinct axes of symmetry L, L_1, L_2, L_3 such that Property (T′) is satisfied for (L,p) w.r.t. the lines L_1, L_2, L_3. If G_L, G_1, G_2, G_3 are the full groups of symmetries about L, L_1, L_2, L_3, respectively, then $G_L \subseteq \langle G_1, G_2, G_3 \rangle$.*

Proof. Put $G = \langle G_1, G_2, G_3 \rangle$. Suppose $M \sim L$ is arbitrary with $M \not{I} p$, and suppose q and q' are distinct arbitrary points on M which are not incident with L. Since Property (T′) is satisfied for the ordered flag (L,p) w.r.t. the lines L_1, L_2, L_3, there is a permutation (i,j,k) of $(1,2,3)$ such that there are lines M_i, M_j, M_k for which $M \in \{M_i, M_k, L\}^{\perp}$, with qIM_i and $q'IM_k$, and $M_j \in \{M_i, M_k, L_j\}^{\perp}$, and such that $M_r \sim L_r$ with $r \in \{i,j,k\}$. For convenience, put $(i,j,k) = (1,2,3)$.

By the considerations above, the following collineations exist:

(1) θ_1 is the symmetry about L_1 which sends $q = M \cap M_1$ to $M_1 \cap M_2$;

(2) θ_2 is the symmetry about L_2 which sends $M_1 \cap M_2$ to $M_2 \cap M_3$;

(3) θ_3 is the symmetry about L_3 which maps $M_2 \cap M_3$ to $q' = M \cap M_3$.

Define the following collineation of $\mathcal{S}$:

$$\theta := \theta_1\theta_2\theta_3.$$

Then θ is an automorphism of $\mathcal{S}$ which is contained in G, and hence θ is an elation about p by Theorem 6.2.1. Also, θ fixes M and maps q onto q'. Now by Theorem 2.3.11, θ is a symmetry about L. It follows now easily that $G_L \subseteq G$ since q and q' were arbitrary. ■

Remark 6.3.2. By Chapter 2 it is sufficient to ask that L is a regular line in order to conclude it is an axis of symmetry.

Note. We emphasize at this point that Property (T) and Property (T′) are purely combinatorial properties which are defined without the use of collineations.

6.4 Divisibility Conditions for Generalized Quadrangles with Symmetry

Theorem 6.4.1. *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s \neq 1 \neq t$, and let L, M and N be three different axes of symmetry incident with the same point p. Then $s|t$ and $\frac{t}{s} + 1|(s + 1)t$.*

Proof. Define G' as the group generated by the symmetries about the lines L, M and N. By Theorem 6.2.1 we have that G' is a group of elations with center p, and the size of G' is s^3. If we consider the permutation group $(P \setminus p^\perp, G')$, then we see that $|G'|$ divides $|P \setminus p^\perp|$, or that $s^3|s^2t$. So t is divisible by s. By Theorem 1.2.1, the theorem now follows. ■

There is a nice corollary:

Corollary 6.4.2. *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t \neq 1$, and suppose that $L_0, L_1, \ldots, L_l$ are $l + 1$ axes of symmetry incident with the point p. Suppose that the group G which is generated by the symmetries about $L_0, L_1, \ldots, L_l$ acts transitively on $P \setminus p^\perp$. Then s and t have the same parity.*

Proof. If s^2t is odd, then s and t are both odd, so suppose that s^2t is even. Since G acts transitively on $P \setminus p^\perp$, we have that s^2t is a divisor of $|G|$. This means that $|G|$ is even. Now suppose that G is generated by the symmetries about $L_0, L_1, \ldots, L_l$, and denote by $G_0, G_1, \ldots, G_l$ the full groups of symmetries about respectively

$L_0, L_1, \ldots, L_l$. So $G = \langle G_0, G_1, \ldots, G_l \rangle$. By Theorem 2.2.2, $G_iG_j = G_jG_i$ for $i \neq j$, $0 \leq i, j \leq l$. If H and H' are subgroups of some group H'', then HH' is a group if and only if $HH' = H'H$. Inductively, if $\{i_0, i_1, \ldots, i_k\} \subseteq \{0, 1, \ldots, l\}$ for a certain k, then $\langle G_{i_0}, G_{i_1}, \ldots, G_{i_k} \rangle = G_{i_0}G_{i_1} \ldots G_{i_k}$. If H and H' (as before) are finite groups, then $|HH'| = \dfrac{|H| \times |H'|}{|H \cap H'|}$ and so by the previous property we obtain that

$$|G| = \frac{s^{(l+1)}}{\prod_{i=0}^{l-1} |[G_0G_1 \ldots G_i] \cap G_{(i+1)}|}. \tag{6.1}$$

Since $|G|$ is even, we have that s necessarily is even (using Equality (6.1)). As we assumed that G acts transitively on $P \setminus p^\perp$, and as $t \geq s$ because $\mathcal{S}$ has regular lines, $|G| \geq |P \setminus p^\perp| = s^2t \geq s^3$, so $l \geq 2$. By Theorem 6.4.1 we conclude that $s|t$. Hence t is even, and the theorem follows. ■

6.5 Property (Sub) and Property (T)

We start this section by introducing *Property (Sub)*.

Property (Sub). Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and suppose M_1, M_2, M_3, M_4 are four distinct lines of $\mathcal{S}$, incident with the same point. Then these four lines satisfy *Property (Sub)* if $\mathcal{S}$ does not have a subGQ of order s containing these four lines.

Observe

Theorem 6.5.1. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, and let M_1, M_2, M_3 and M_4 be four distinct axes of symmetry through the point p of $\mathcal{S}$ such that there is a line $M \in \{M_1, M_2, M_3, M_4\} = \mathcal{M}$ for which Property (T) is satisfied for the ordered flag (M,p) w.r.t. the lines of $\mathcal{M} \setminus \{M\}$. Then these four lines also satisfy Property (Sub).*

Proof. Suppose that the lines M_1, M_2, M_3 and M_4 are contained in a proper subGQ $\mathcal{S}'$ of order s and suppose that $M = M_4$ is such that (M,p) satisfies Property (T) w.r.t. the lines M_1, M_2, M_3. Consider lines L with $L \sim M_4$ and L' with $L' \sim M_1$, both contained in $\mathcal{S}'$ and both not incident with p, such that $L \not\sim L'$. The lines M_i, $i = 1, 2, 3, 4$, are axes of symmetry in the quadrangle $\mathcal{S}'$, so they are regular. Now, each triad of lines in $\mathcal{S}'$ which contains one of those lines is centric (it is easily seen that a pair of lines $\{U, V\}$ in a GQ of order s, $s > 1$, is regular if and only if each triad containing U and V is centric, see Theorem 1.4.2(ii)). This leads to a contradiction. ■

Of course, Theorem 6.5.1 could be 'restated' more generally as follows: *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and let M_1, M_2, M_3 and M_4 be four distinct lines through the point p of $\mathcal{S}$ so that M_1, M_2 and M_3 are axes of symmetry, and so that Property (T) is satisfied for the ordered flag (M_4, p) w.r.t. M_1, M_2, M_3. Then these four lines satisfy Property (Sub).*

Theorem 6.5.2. *Suppose that* $(\mathcal{S}^{(p)}, G) = (P, B, I)$ *is a TGQ of order* (s, s^2), $s > 1$, *and suppose that* $\mathcal{S}$ *has a subquadrangle* $\mathcal{S}'$ *of order* s *which contains the point* p. *Then there exist four lines incident with* p, *such that* G *is the group generated by the symmetries about these lines.*

Proof. The quadrangle $\mathcal{S}'$ is a TGQ of order s with translation point p, and so there exist three lines in $\mathcal{S}'$ incident with p, such that the translation group G' of $\mathcal{S}'$ is generated by all symmetries about these three lines. Consider a line L of $\mathcal{S}$ incident with p and not contained in $\mathcal{S}'$. This line intersects $\mathcal{S}'$ only in p. Take an arbitrary line M of $\mathcal{S}$ intersecting L and not incident with p. This line is not contained in $\mathcal{S}'$ as L is not a line of $\mathcal{S}$, and thus it intersects $\mathcal{S}'$ in just one point. Since L is an axis of symmetry, the group G_L of symmetries about L acts transitively on the points of $M \setminus \{L \cap M\}$; it follows now that every G'-orbit in $P \setminus p^\perp$ intersects M in exactly one point. Also, there follows that $G'' = \langle G', G_L \rangle$ has size s^4. Since it is now clear that $G'' = G$, the theorem is proved. ■

Note. It is clear that four is the minimal number of lines such that the translation group of a TGQ of order (s, t), $1 < s \neq t$, is generated by the symmetries about these lines, see e.g. the proof of Theorem 6.2.2.

Suppose that $\mathcal{S}^{(p)}$ is a thick TGQ of order (s, t) which satisfies Property (T) for some ordered flag (L, p) w.r.t. the lines L_1, L_2, L_3. Then by Theorem 6.2.2 we have that $t = s^2$. Now suppose that $\mathfrak{C}$ is the class of all thick TGQ's $\mathcal{S}^{(p)}$, where p denotes the translation point, for which the following condition holds:

(C) *If* L_1, L_2, L_3, L_4 *are distinct lines through* p *for which Property (Sub) is satisfied, then there is a line* $L \in \{L_1, L_2, L_3, L_4\} = \mathcal{L}$ *such that* $\mathcal{S}^{(p)}$ *satisfies Property (T) for the ordered flag* (L, p) *w.r.t. the lines of* $\mathcal{L} \setminus \{L\}$.

Note that by Theorem 6.5.1, $\mathfrak{C}$ is exactly the class of TGQ's for which Property (T) and Property (Sub) are 'equivalent' (with respect to certain lines incident with p). Of course, each TGQ of order s is an element of $\mathfrak{C}$. Suppose that $\mathcal{S}^{(p)} \in \mathfrak{C}$ is of order (s, t), $s \neq t$, and suppose that $\mathcal{S}^{(p)}$ does *not* satisfy Property (T) for some ordered flag (p, L) w.r.t. some three distinct lines through p and different from L. It is easy to observe that such a 4-tuple of lines always exists for $\mathcal{S}^{(p)}$. Then Property (Sub) does not hold, and so there is a subGQ of order s containing those four lines. It follows that $t = s^2$ by Theorem 1.3.1, but moreover, we also have the following essential observation for $\mathcal{S}^{(p)}$: for each four distinct lines U, V, W, Z through p, we have that

(i) either they are contained in a subGQ of $\mathcal{S}^{(p)}$ of order s, or

(ii) for each $O \in \{U, V, W, Z\}$, (O, p) satisfies Property (T) w.r.t. $\{U, V, W, Z\} \setminus \{O\}$.

By Theorem 2.4.3, there now readily follows that $\mathcal{S}^{(p)} \cong T_3(\mathcal{O})$, where $\mathcal{O}$ is an ovoid of $\mathbf{PG}(3, s)$, see the beginning of Section 6.6 for more details.

We hence have the following result.

Theorem 6.5.3. *Let $\mathcal{S}^{(p)}$ be a thick element of order (s,t) of $\mathfrak{C}$. Then we have one of the following possibilities:*

(i) $\mathcal{S}^{(p)}$ *is a TGQ of order s without any further restrictions;*

(ii) $\mathcal{S}^{(p)} \cong T_3(\mathcal{O})$*, where $\mathcal{O}$ is an ovoid in* $\mathbf{PG}(3,s)$ *(so if s is odd, $\mathcal{S}^{(p)} \cong \mathcal{Q}(5,s)$).*

In particular, there always exist four lines through p which 'determine' the translation group, in the sense that the translation group is generated by the symmetries about those lines. ■

In view of the fact that we want to study the TGQ's which satisfy Property (T) for a certain ordered flag (through the translation point) w.r.t. certain lines, the latter result seems not to be agreeable in that sense, that it does not even 'include' general TGQ's $\mathcal{S} = T(\mathcal{O})$ with $\mathcal{O}$ good at some element. To that end, we will obtain a more general classification result in the next section. A combination of Property (T) and Property (T′) appears to be the right choice.

6.6 Property (T), Property (T′) and Property (G)

6.6.1 Property (T), Property (T′) and Eggs

We now situate Property (T) in the theory of TGQ's, as follows.

Theorem 6.6.1. *Let $\mathcal{S} = T(n,m,q) = T(\mathcal{O})$ be a TGQ of order (q^n,q^m) for which there exist four distinct elements π_i, $i = 1,2,3,4$, of $\mathcal{O} = \mathcal{O}(n,m,q)$ which generate a $(4n-1)$-space. Then $\mathcal{S}$ is a GQ of order (q^n,q^{2n}) and $\mathcal{S}$ satisfies Property (T) for every ordered flag $(\pi_r,(\infty))$, $r \in \{1,2,3,4\}$, w.r.t. π_i,π_j,π_k, where $\{i,j,k\} = \{1,2,3,4\} \setminus \{r\}$.*

Proof. As usual, we represent $\mathcal{S} = T(\mathcal{O})$ in $\mathbf{PG}(2n+m,q)$, where $\mathcal{O}$ generates a $\mathbf{PG}(2n+m-1,q) \subseteq \mathbf{PG}(2n+m,q)$. Since $4n-1 \leq 2n+m-1$, $\mathcal{S}$ is a GQ of order (q^n,q^{2n}). Now, let $L_k \sim \pi_k$, $k = 1,2,3,4$, be lines of $\mathcal{S}$ such that $\mathcal{V}_3 = \{L_1,L_2,\pi_3\}$ and $\mathcal{V}_4 = \{L_1,L_2,\pi_4\}$ are centric triads with $L_3 \in \mathcal{V}_3^{\perp}$ and $L_4 \in \mathcal{V}_4^{\perp}$. Note that we assume that $L_1 \not\sim L_2$. It follows that the n-spaces L_3 and L_4 intersect the $(2n+1)$-dimensional space generated by the n-spaces L_1 and L_2 in a space of dimension at least 1, thus, if π is the space generated by the n-spaces L_k, $1 \leq k \leq 4$, then π has dimension at most $4n-1$. If we intersect π with $\mathbf{PG}(4n-1,q)$, then we obtain a space of dimension at most $4n-2$ which contains the π_k's, a contradiction. Hence, for every ordered flag $(\pi_r,(\infty))$, $r = 1,2,3,4$, Property (T) is satisfied w.r.t. π_i,π_j,π_k, where $\{i,j,k\} = \{1,2,3,4\} \setminus \{r\}$. ■

The following theorem is a converse of Theorem 6.6.1.

Theorem 6.6.2. *Suppose $\mathcal{S}^{(\infty)}$ is the thick TGQ of order (q^n, q^{2n}) which corresponds to the generalized ovoid $\mathcal{O}$ of $\mathbf{PG}(4n-1,q)$. If $LI(\infty)$ is a line of $\mathcal{S}^{(\infty)}$ such that Property (T) is satisfied for the ordered flag $(L,(\infty))$ with respect to the lines L_1, L_2, L_3 through (∞) (where $|\{L, L_1, L_2, L_3\}| = 4$), and if π, respectively π_i, is the element of $\mathcal{O}$ which corresponds to L, respectively L_i, $i = 1,2,3$, then $\langle \pi, \pi_1, \pi_2, \pi_3 \rangle = \mathbf{PG}(4n-1,q)$.*

Proof. Immediate from the proof of Theorem 6.2.2 and the interpretation in the projective model $T(n, 2n, q)$. ■

Theorem 6.6.3. *A TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ which is good at an element $\pi \in \mathcal{O}$ satisfies Property (T) for the ordered flag $(\pi, (\infty))$ w.r.t. any three distinct lines through (∞) which are different from π, and which generate a $(3n-1)$-space not containing π.*

Proof. If $\mathcal{O}$ is good at some element π, and if the three lines L_1, L_2, L_3 are as above, then π, L_1, L_2, L_3 generate a $\mathbf{PG}(4n-1,q)$. ■

Thus, if $\mathcal{S} = T(\mathcal{O})$ is a TGQ for which $\mathcal{O}$ has a good element, then there are always four lines such that the translation group is generated by the symmetries about these lines.

Corollary 6.6.4. *The $T_3(\mathcal{O})$ of Tits satisfies Property (T) for each ordered flag $(L,(\infty))$ w.r.t. any three distinct lines through (∞), all different from L, and so that these four lines are in general position.*

Proof. We have that $\mathcal{O}$ is good at every point by Theorem 2.4.3. ■

Lemma 6.6.5. *Suppose that $(\mathcal{S}^{(p)}, G)$ is a thick TGQ of order (s,t), $t \geq 3$. Then $\mathcal{S}$ is of order s if and only if there are distinct lines L_1Ip, L_2Ip, L_3Ip such that for every other line LIp, with G_i the full group of symmetries about L_i and G_L the full group of symmetries about L, $i = 1,2,3$, the group $\langle G_1, G_2, G_3, G_L \rangle$ has size s^3.*

Proof. It is clear that a TGQ $(\mathcal{S}^{(p)}, G)$ of order s, $s > 1$, has the desired property, since $|G| = s^3$ and since by Theorem 2.3.3, G is generated by the symmetries about three arbitrary distinct lines through p.
Let $(\mathcal{S}^{(p)}, G)$ be a TGQ, and suppose that the required conditions are satisfied. Suppose that G_i is the full group of symmetries about the line L_i, with $i \in \{1, 2, \ldots, t+1\}$ and L_iIp. If $t = 3$, then for any thick TGQ of order (s,t) we have $s = 3$ and so $|G| = s^3$. Hence let $t \geq 4$. Define the group H_j as $H_j = \langle G_1, G_2, G_3, \ldots, G_j \rangle$ with $j \in \{4, 5, \ldots, t+1\}$. Then $|H_4| = s^3$. Considering that a group generated by the symmetries about three concurrent axes of symmetry is a group of elations of order s^3 about their intersection point (by Theorem 6.2.1), we have that $H_4 = \langle G_1, G_2, G_3 \rangle$. But $H_5 = \langle H_4, G_5 \rangle$, so $H_5 = H_4$, and thus also $H_4 = H_5 = \ldots = H_{t+1}$. Since $H_{t+1} = G$, we have $|G| = s^3$, and hence $\mathcal{S}$ is of order s. ■

The following theorem implies that Property (T′) is a characteristic property for TGQ's of order s.

Theorem 6.6.6. *Suppose $(\mathcal{S}^{(p)}, G)$ is a thick TGQ of order (s,t), $t \geq 3$. Then $\mathcal{S}$ is of order s if and only if there is a line $L\mathrm{I}p$ such that $\mathcal{S}$ satisfies Property (T′) for the ordered flag (L,p) w.r.t. every three distinct lines L_1, L_2, L_3 through p which are different from L.*

Proof. Immediately by Lemma 6.6.5 and Theorem 6.3.1. ■

Now observe

Theorem 6.6.7. *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s,t) with $s \neq 1 \neq t$, such that there is a line $L\mathrm{I}p$ so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T′) is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3, and suppose that there is at least one 3-tuple (M,N,U) of distinct lines incident with p, such that (L,p) satisfies Property (T) w.r.t. (M,N,U). Then $T(\mathcal{O})$ is good at its element L.*

Proof. Since there is at least one 3-tuple (M,N,U) as above such that (L,p) satisfies Property (T) w.r.t. (M,N,U), we have by Theorem 6.2.2 that $t = s^2$. By Theorem 6.6.2, the definitions of Property (T) and Property (T′), Lemma 6.6.5 and Theorem 6.6.6, there follows that the generalized ovoid $\mathcal{O}$ satisfies the condition that for every three distinct lines L_1, L_2, L_3 through p and different from L, the projective space generated by the four corresponding elements of $\mathcal{O}$ either is a $\mathbf{PG}(4n-1,q)$ or a $\mathbf{PG}(3n-1,q)$. Thus every $(3n-1)$-space which is generated by the element π of $\mathcal{O}$ which corresponds to L and two other elements of $\mathcal{O}$, has the property that it either is disjoint with any other element of $\mathcal{O}$ or completely contains it. Suppose that we denote the elements of $\mathcal{O}$ by $\pi, \pi^1, \ldots, \pi^{q^{2n}}$, where π corresponds to L, and fix for instance π^i, i arbitrary. Then all the (different) $(3n-1)$-spaces of $\mathbf{PG}(4n-1,q)$ which contain π, π^i and an element of $\mathcal{O} \setminus \{\pi, \pi^i\}$ intersect two by two in $\pi\pi^i$ and cover $\mathbf{PG}(4n-1,q)$. By the preceding remarks, every element of $\mathcal{O} \setminus \{\pi, \pi^i\}$ is completely contained in one of these $(3n-1)$-spaces, and is disjoint with any other of these $(3n-1)$-spaces. Since the number of these spaces is q^n+1, every of these $(3n-1)$-spaces contains exactly q^n+1 elements of $\mathcal{O}$. The theorem now follows since i was arbitrary. ■

Note. It is also possible to prove Theorem 6.6.7 with the use of 8.7.2 of FGQ.

Theorem 6.6.8. *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s,t) with $s \neq t$, such that there is a line $L\mathrm{I}p$ so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T′) is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3. Then $t = s^2$ and the translation dual $\mathcal{S}^* = T(\mathcal{O}^*)$ of $\mathcal{S}^{(p)}$ satisfies Property (G) at the flag (p', L'), where (p', L') corresponds to (p, L).*

Proof. Immediate by Theorem 6.6.7 and Theorem 2.5.1. ■

There is a very nice corollary of Theorem 2.7.4 and Theorem 6.6.7.

Corollary 6.6.9. *Suppose that $\mathcal{S} = \mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s,t) with $s \neq t$ and s odd, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T′) is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3. Then $t = s^2$ and the translation dual $\mathcal{S}^* = T(\mathcal{O}^*)$ of $\mathcal{S}$ is the point-line dual of a flock GQ of order (s^2, s).*

Proof. By Theorem 6.6.8 we have that $t = s^2$, and the translation dual $\mathcal{S}^*$ of $\mathcal{S}$ satisfies Property (G) at its flag (p', L') which corresponds with (p, L). By Theorem 2.7.4, the proof is complete. ∎

6.6.2 Flocks, Property (T) and Property (T′): Classification

We are now able to classify the thick TGQ's for which there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T′) is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3, as follows.

Theorem 6.6.10. *Suppose that $\mathcal{S}^{(p)}$ is a thick TGQ of order (s,t), $s \neq 1 \neq t$, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T′) is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3. Then we have the following classification.*

(i) *$s = t$ and $\mathcal{S}$ is a TGQ with no further restrictions.*

(ii) *$t = s^2$, s is an even prime power, and $\mathcal{O}$ is good at its element π which corresponds to L, where $\mathcal{S} = T(\mathcal{O})$. Also, $\mathcal{S}$ has precisely $s^3 + s^2$ subGQ's of order s which contain the line L, and if one of these subquadrangles is classical, i.e. isomorphic to the GQ $\mathcal{Q}(4,s)$, then $\mathcal{S}$ is classical, that is, isomorphic to the GQ $\mathcal{Q}(5,s)$.*

(iii) *$t = s^2$ and $s = q^n$, q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S}^{(p)}$, with $q \geq 4n^2 - 8n + 2$, and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock.*

(iv) *$t = s^2$ and $s = q^n$, q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S}^{(p)}$, with $q < 4n^2 - 8n + 2$, and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

Proof. If for every three distinct lines L_1, L_2, L_3 through p and different from L, Property (T′) is always satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3, then by Lemma 6.6.6, $\mathcal{S}$ is a TGQ of order s with no further restrictions. Suppose this is not the case. Then there is a 3-tuple (M, N, U) of distinct lines through p such that Property (T) is satisfied for the ordered flag (L,p) w.r.t. M, N, U ($L \notin \{M, N, U\}$). Hence by Theorem 6.2.2 there follows that $t = s^2$, and if $\mathcal{S} = T(\mathcal{O})$, then by Theorem 6.6.7, $\mathcal{O}$ is good at its element π which corresponds to L. Suppose that

s is even. Then (ii) follows from Theorem 2.5.2 and Theorem 2.5.4. Next suppose that s is odd. By Theorem 2.7.4, $T^*(\mathcal{O})$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. Then (iii) and (iv) follow from Theorem 3.4.3. ■

6.7 The General Problem

We start this section with the following result.

Theorem 6.7.1 (FGQ, 9.4.2). *Suppose that $L_0, L_1, \ldots, L_r$, $r \geq 1$, are $r+1$ lines incident with a certain point p in the GQ $\mathcal{S} = (P, B, I)$ of order (s,t), $s \neq 1 \neq t$. Suppose that O is the set of points different from p, which are on the lines of $L_0, L_1, \ldots, L_r$, and denote $P \setminus p^\perp$ by Ω. Suppose that G is a group of elations with center p, and suppose G has the property that, if M is an arbitrary line which intersects O in one point m, then G acts transitively on the points of Ω lying on M. If $r > t/s$, then G acts transitively — and hence also regularly — on the points of Ω.* ■

Corollary 6.7.2. (1) *Suppose that $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s,t), $s,t \neq 1$, and suppose $k > t/s$, $k \in \mathbb{N}$. Then the translation group G is generated by the symmetries about $k+1$ arbitrary lines through the translation point p.*

(2) *Let $\mathcal{S} = (P, B, I)$ be a thick GQ of order (s,t) with the property that there is a point p incident with at least $s+2$ distinct axes of symmetry, and suppose that the group G generated by all symmetries about $s+2$ axes of symmetry through p is a group of elations. Then $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

(3) *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, with the property that there is a point p incident with at least $s+2$ distinct axes of symmetry, and define G to be the group generated by all symmetries about these lines. Suppose that, for each point $q \in P \setminus p^\perp$, $|\{p,q\}^{\perp\perp}| = 2$. Then $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

(4) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $t > s^2/2$ and $s \neq 1$, with x a point incident with $r+1$ axes of symmetry, $r \geq s+1$. If G is the group generated by all symmetries about these $r+1$ lines, then $(\mathcal{S}^{(x)}, G)$ is a TGQ.*

Proof. The first assertion is immediate. Consider Case (2). From the inequality of Higman follows that $t/s \leq s$, hence $s+1 > t/s$. So, the conditions of Theorem 6.7.1 for the group G and the $s+2$ lines $L_i I p$ are satisfied. Hence, the group G acts regularly on the points of $P \setminus p^\perp$, and $(\mathcal{S}^{(p)}, G)$ is an EGQ. There are at least two regular lines through the elation point p, and by Theorem 2.3.15 (or Theorem 2.3.1) the proof is complete. Suppose we are in Case (3). By Theorem 2.1.3(v) the conditions of Theorem 6.7.1 are satisfied, which implies that $(\mathcal{S}^{(x)}, G)$ is an EGQ. Theorem 2.3.15 (or Theorem 2.3.1) finishes the proof. It remains to prove (4). If $t > s^2/2$, then each span of non-collinear points containing p has size 2 by Theorem 1.4.5. Whence the result (apply Part (3)). ■

Note. The assumption '$t > s^2/2$' in Corollary 6.7.2(4) seems a bit artificial; one would like to demand, for instance, that $t \geq s^2/2$ — probably the only thick GQ which satisfies $t = s^2/2$ is $\mathcal{Q}(4,2)$. However, the standard divisibility condition ($s+t$ divides $st(st+1)$) for GQ's does not exclude all hypothetical GQ's of order $(s, s^2/2)$ with $s > 2$ (for instance, put $s = 10$).

We are ready to obtain

Theorem 6.7.3. *Suppose $\mathcal{S} = (\mathcal{S}^{(p)}, G)$ is a thick TGQ of order (s,t), and let $\mathbf{GF}(q)$ be the kernel of the TGQ. Next, suppose that $L_1, L_2, \ldots, L_{t+1}$ are the lines incident with p, and let G_i be the group of all symmetries about the line L_i, $i \in \{1, 2, \ldots, t+1\}$. Define k as the smallest number such that $G = \langle G_{i_1}, G_{i_2}, \ldots, G_{i_{(3+k)}}\rangle$, with $\{i_1, i_2, \ldots, i_{(3+k)}\} \subseteq \{1, 2, \ldots, t+1\}$. Then we have the following inequality:*

$$k \leq \log_q \frac{t}{s}.$$

Proof. Denote the groups $\langle G_{i_1}, G_{i_2}, G_{i_3}\rangle$ and $\langle G_{i_1}, G_{i_2}, G_{i_3}, \ldots, G_{i_{(j+3)}}\rangle$ respectively by G'_0 and G'_j, with $j \in \{1, 2, \ldots, k\}$. By Theorem 6.2.1 we have that $|G'_0| = s^3$. Since k is defined as a minimum, we have the following strict chain of groups:

$$G'_0 \leq G'_1 \leq \ldots \leq G'_k = G.$$

Now fix a point $y \in P \setminus p^\perp$, where $\mathcal{S} = (P, B, I)$. The groups G'_i are all groups of elations about p, and hence for the G'_i-orbits $(G')^*_i$ which contain y, we have that $|(G')^*_i| = |G'_i|$, $i = 0, 1, \ldots, k$, and that

$$(G')^*_0 \subset (G')^*_1 \subset \ldots \subset (G')^*_k = P \setminus p^\perp.$$

The TGQ $\mathcal{S}$ is a $T(\mathcal{O})$ for some egg $\mathcal{O}$ in $\mathbf{PG}(2n+m-1, q) \subseteq \mathbf{PG}(2n+m, q)$, where $\mathbf{GF}(q)$ is the kernel of the TGQ. If we interpret the aforementioned strict chain of orbits in $\mathbf{PG}(2n+m-1, q)$, then we obtain a strict chain of affine spaces over $\mathbf{GF}(q)$:

$$\mathbf{AG}'_0 = \mathbf{AG}(3n, q) \subset \mathbf{AG}'_1 \subset \ldots \subset \mathbf{AG}'_k = \mathbf{AG}(2n+m, q),$$

and we have that $|\mathbf{AG}'_j| \geq q|\mathbf{AG}'_{j-1}|$ for every $j \in \{1, 2, \ldots, k\}$. This implies that $|G| \geq q^k s^3$. Since $|G| = s^2 t$, the theorem follows. ■

This leads to one of the main theorems of this chapter, which is a considerable improvement of the best known (general) result.

Theorem 6.7.4. *Let $(\mathcal{S}^{(p)}, G)$ be a TGQ of order (s,t), $1 \neq s \neq t \neq 1$, with $(s,t) = (q^{na}, q^{n(a+1)})$, where $\mathbf{GF}(q)$ is the kernel of the TGQ and where a is odd. If $k+3$ is the minimal number of distinct lines through p such that G is generated by the symmetries about these lines, then*

$$k \leq n.$$

■

Remark 6.7.5 (An Alternative Approach for $T_3(\mathcal{O})$). Suppose $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s,t), $s \neq 1 \neq t \neq s$. Then the kernel $\mathbb{K}$ of the TGQ is isomorphic to $\mathbf{GF}(s)$ if and only if $\mathcal{S}$ is a $T_3(\mathcal{O})$ with $\mathcal{O}$ some ovoid of $\mathbf{PG}(3,s)$, see Theorem 2.4.3. From Theorem 6.7.4 immediately follows that there are four distinct lines incident with p such that G is generated by all symmetries about these four lines. Hence, the knowledge of the size of the kernel is already sufficient to completely solve Problem (2) from the introduction for the $T_3(\mathcal{O})$ of Tits.

Part (1) of the following is an analogue of Theorem 6.7.4 in a more general context. Part (2) generalizes the fact that if $\mathcal{S}$ is a (thick) TGQ of order (s,t), then s and t are powers of the same prime.

Theorem 6.7.6 (Structure Theorem). (1) *Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and let x be a point of $\mathcal{S}$ incident with $r+1$ axes of symmetry $L_0, L_1, \ldots, L_r$. Suppose G is the group generated by all symmetries about the lines L_i, $0 \leq i \leq r$. We denote the full group of symmetries about L_i by G_i, $0 \leq i \leq r$. Define k as the smallest natural number such that $|G| \leq s^3k$, and suppose $r \geq 2$. Then there are at least $m = r-2-\log_p k$ groups of $\{G_0, G_1, \ldots, G_r\}$ which are abelian, and G is generated by the symmetries about at most $3+\log_p k$ elements of $\{L_0, L_1, \ldots, L_r\}$. Here p is the smallest prime number dividing s.*

(2) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $t > s^2/2$ and $s \neq 1$, and let x be a point which is incident with $r+1$ axes of symmetry $L_0, L_1, \ldots, L_r$, $r \geq 2$, such that the following condition is satisfied.*

- *If G_i is the full group of symmetries about L_i, $i = 0, 1, \ldots, r$, and $H_i = \langle G_j \parallel j \neq i, 0 \leq j \leq r\rangle$, then $G_i \not\subseteq H_i$.*

If $r+1 \geq 3+\log_p \frac{t}{s}$, then $r+1 = 3+\log_p \frac{t}{s}$ and $(\mathcal{S}^{(x)}, G)$ is a TGQ. Here p is the smallest prime number dividing s, and $G = \langle G_0, G_1, \ldots, G_r\rangle$. Also, $\frac{t}{s}$ is a power of p, and s and t are powers of p if $t = s^2$.

(3) *Suppose that $\mathcal{S} = (P, B, I)$ is a thick GQ of order (s,t), with x a point which is incident with at least $r+1$ axes of symmetry, $L_0, L_1, \ldots, L_r$, $r \geq 2$. Let G_i be the full group of symmetries about L_i, $i = 0, 1, \ldots, r$. Define $G = \langle G_i \parallel i = 0, 1, \ldots, r\rangle$, and put $H_i = \langle G_j \parallel j \neq i; j \in \{0, 1, \ldots, r\}\rangle$. Suppose that G_* is an arbitrary G-orbit in $P \setminus x^\perp$. If for all $i = 0, 1, \ldots, r$, H_i acts transitively on the points of G_*, then G acts regularly on the points of G_*, G is abelian and G is a group of elations about x. The same properties then hold for every G-orbit in $P \setminus x^\perp$.*

Proof. (1) By Theorem 6.2.1 we have that s^3 divides $|G|$, and hence $|G| = s^3k$ (where k is as above). Suppose $r+1-m$ is the minimal number of groups of $\{G_0, G_1, \ldots, G_r\}$ which generate G. Then we have by the proof of Theorem 6.7.3 (cf. the strict chain of groups) that

$$s^3p^{r-2-m} \leq |G| = s^3k,$$

and hence $p^{r-2-m} \leq k$, thus $m \geq r - 2 - \log_p k$.
Next, suppose that $L_0, L_1, \ldots, L_r$ are axes of symmetry through p, indexed in such a way that $G = \langle G_j \parallel m \leq j \leq r\rangle$ (recall that G is generated by all symmetries about $r + 1 - m$ axes of symmetry incident with x), and define H_i as $H_i = \langle G_j \parallel j \neq i,\ 0 \leq j \leq r\rangle$, $i \in \{0, 1, \ldots, r\}$. Then we have that $G = H_iG_i = G_iH_i$ for every i, $i \in \{0, 1, \ldots, r\}$, since symmetries about distinct concurrent lines commute, and if $j \in \{0, 1, \ldots, m-1\}$, it follows that $G = H_jG_j = H_j$ $(G_j \leq H_j = G)$. Also, as symmetries about distinct concurrent lines commute, every group G_j, with $j \in \{0, 1, \ldots, m-1\}$, is abelian, see the proof of [91, 8.3.1] (this is Theorem 2.1.4). This proves the first part of the result.

(2) From Part (1) and Theorem 6.4.1 follows that $|G| \geq p^{r-2}s^3$. Since $r + 1 \geq 3 + \log_p \frac{t}{s}$, we have that $|G| \geq s^2t$. Since $t > s^2/2$, it follows by Theorem 2.1.3 that G is a group of elations with center x, hence $|G| = s^2t$ and $r = 2 + \log_p \frac{t}{s}$. The equality implies that $\frac{t}{s}$ is a power of p, and in particular, if $t = s^2$, then s is a power of p, and then also t. Since $|G| = s^2t$, $(\mathcal{S}^{(x)}, G)$ is an EGQ, and by, e.g., Theorem 2.3.15 and the fact that $r \geq 1$, we also know that $\mathcal{S}^{(x)}$ is a TGQ.

(3) Suppose $|G_*| = m$; then $m \geq s^3$ by Theorem 6.2.1, and so $|G| = mk$, with $k = |G_y|$ for an arbitrary point $y \in G_*$. In the following y will be fixed, as well as $i \in \{0, 1, \ldots, r\}$. We have $|H_i| = mk'$ with $k' = |(H_i)_y|$, and clearly that $k'|k$, say $k = k'n$. So we have that:

$$|G| = mk = |G_iH_i| = \frac{|G_i| \times |H_i|}{|G_i \cap H_i|} = \frac{smk'}{|G_i \cap H_i|}. \tag{6.2}$$

Hence $s = n|G_i \cap H_i|$, and so $n|s$. Suppose now that p is a prime which divides n; then there exists a $\theta \in G_y$ of order p. Suppose M is a line through y meeting L_i in x_i. The orbits of $\langle\theta\rangle$ on M (seen as a point set) are cycles of length p or length 1, and since p is a divisor of n and of s, we have that there are at least $p+1$ points incident with M which are fixed by θ. By Theorem 1.3.4, θ has to be the identity, since $s \leq t$ (as an axis of symmetry is a regular line). It follows that $n = 1$, and that $H_i = G$. Every two symmetries about distinct concurrent lines commute, and since we proved that $G_jH_j = H_j = G$, with $j = 0, 1, \ldots, r$, G_j is commutative for every j. Hence G is abelian, and since G acts transitively on G_*, G also acts regularly on G_*. Finally, suppose that H_* is an arbitrary G-orbit in $P \setminus x^\perp$. Since G is abelian and since G acts transitively on this orbit, we can again conclude that G acts regularly on H_*. This completely proves the assertion. ■

Theorem 6.7.7. *Suppose $\mathcal{S} = (P, B, I)$ is a generalized quadrangle with parameters (s, t), $s \neq 1 \neq t$, and assume that p is a point which is incident with at least three axes of symmetry. Also, suppose that G is the group generated by the symmetries about* every *axis of symmetry through p. Suppose G_* is an arbitrary G-orbit of the permutation group $(P \setminus p^\perp, G)$. Now define the incidence structure $\mathcal{S}'(G_*) = \mathcal{S}' = (P', B', I')$ as follows.*

- *The* POINTS *of P' are of three types:*
  (1) *the point p;*
  (2) *the points of G_*;*
  (3) *any point which is incident with an axis of symmetry through p.*
- *We have two types of* LINES*:*
  (a) *the axes of symmetry through p;*
  (b) *the lines of $\mathcal{S}$ which intersect a line of Type (a) and contain at least one point of G_*.*
- *The* INCIDENCE RELATION *$I' \subseteq I$ is the restriction of I to $(P' \times B') \cup (B' \times P')$.*

Then we have the following properties.

(1) *There are constants l and k such that any point of the first two types is incident with $l+1$ lines of $\mathcal{S}'$, and every point of the last type is incident with $k+1$ lines.*

(2) *A line of $\mathcal{S}'$ contains $s+1$ points of $\mathcal{S}'$.*

(3) *$|G_*| = s^2k$.*

(4) *k is divisible by s, and in particular we have that $s \leq k$. Also, $l \leq k$.*

(5) *The number of points of $\mathcal{S}'$ is $ks^2 + (l+1)s + 1$, and the number of lines of $\mathcal{S}'$ is $(l+1)(sk+1)$.*

Proof. (1) Let L be an arbitrary line of $\mathcal{S}'$ through p, and consider an arbitrary point qIL, $q \neq p$. Suppose that q is incident with $k+1$ lines of $\mathcal{S}'$. Since G acts transitively on the points of $L \setminus \{p\}$ (p is incident with at least one axis of symmetry), and since G_* is fixed by G, we can conclude that every point of $L \setminus \{p\}$ is incident with $k+1$ lines of $\mathcal{S}'$. Next consider an arbitrary line L' of $\mathcal{S}'$, $L'Ip$, such that $L' \neq L$, and an arbitrary point $q'IL'$, $q' \neq p$ (so q' is a point of $\mathcal{S}'$). If $k'+1$ is the number of lines of $\mathcal{S}'$ incident with q', then we can easily see that $k' \geq k$ (the $k+1$ lines through q' of $\mathcal{S}$ meeting the $k+1$ lines through q of $\mathcal{S}'$ are also lines of $\mathcal{S}'$), and conversely we have that $k \geq k'$. It follows that there exists a $k \in \mathbb{N}$ such that each point of $\mathcal{S}'$ of Type (3) is incident with $k+1$ lines of $\mathcal{S}'$. Suppose that p is incident with $l+1$ lines of $\mathcal{S}'$, and consider a point p' of G_*. From the definition of $\mathcal{S}'$, we immediately see that p' is also incident with $l+1$ lines of $\mathcal{S}'$. This proves Part (1) of the theorem.

(2) Immediate by the definition of $\mathcal{S}'$.

(3) Consider an arbitrary line $L \in B'$ of Type (a); then each point of G_* is incident with a unique line of $\mathcal{S}'$ (of Type (b)) which is concurrent with L. The statement follows easily by Part (1).

(4) If G' is a group generated by all symmetries about three distinct axes of symmetry through p, then $|G'| = s^3$ by Theorem 6.2.1, and G' is a group of elations with center p. Thus s^3 is a divisor of $|G_*| = s^2k$. Now suppose that M is a fixed line of $\mathcal{S}'$ of Type (b). Counting the number of pairs (q, N), with $q \not{\mathrm{I}} M$ a point of G_* and $q\mathrm{I}N \neq M \sim N$, N a line of Type (b), we obtain that

$$s + sl(s-1) + (k-1)s \leq |G_*| = s^2k,$$

or that

$$sl - l \leq sk - k.$$

Hence Part (4) of the theorem follows.

(5) The number of points, respectively lines, of $\mathcal{S}'$ follows immediately by (1) and (2). ∎

Remark 6.7.8. (i) The geometries $\mathcal{S}'$ as above are not always subGQ's. If $\mathcal{S}$ is, for example, a TGQ with $s \neq t$, then this is only (always) the case if and only if $\mathcal{S}$ is isomorphic to a $T_3(\mathcal{O})$ of Tits.

(ii) In our Master Thesis [141], we axiomatized the incidence structures of Theorem 6.7.7 to what we then called '(l, k)-partial quadrangles'. In some cases they were shown to be generalized quadrangles. In Section 6.10 (see also Section 6.9.1), we will search for a slightly different definition for the natural geometries which arise from generalized quadrangles which have some concurrent axes of symmetry. These geometries will then be investigated in Appendix A.

We are now ready to state the following answer to Problem (2) of the introduction.

Theorem 6.7.9. *Suppose $\mathcal{S} = (P, B, \mathrm{I})$ is a GQ of order (s,t), $t \geq s \neq 1$, and let p be a point of $\mathcal{S}$ which is incident with more than $t-s+2$ axes of symmetry. Then $\mathcal{S}^{(p)}$ is a translation generalized quadrangle. If $s \neq t$, and G is the group generated by the symmetries about $t-s+2$ arbitrary axes of symmetry through p, then G is the translation group.*

Proof. If $s = t$, then the theorem follows from Theorem 2.3.3, so suppose that $s \neq t$. Consider $r+1$ axes of symmetry through p, with $r = t-s+1$, and suppose that G_* is an arbitrary G-orbit in $P \setminus p^\perp$, with G as above. Since p is incident with at least three axes of symmetry, we have by Theorem 6.4.1 that t is divisible by s, and since $r \geq 2$, we can use Theorem 6.7.7 (in the following we use the same notations as in the proof of that theorem). If $\mathcal{S}'$ is the incidence structure associated to G_* as defined in Theorem 6.7.7, then it follows that $k \geq t-s+1$ since $k \geq l \geq r$. However, by Theorem 6.7.7, $s|k$, and we already remarked that $s|t$. Since $k \leq t$, necessarily $k = t$, thus $|G_*| = s^2k = s^2t$. So G acts transitively on $P \setminus p^\perp$. Since the $t-s+2$ axes were arbitrary chosen, the conditions of Theorem 6.7.6(3) hold. Hence we can conclude that the abelian group G acts regularly on $P \setminus p^\perp$, and the theorem follows. ∎

Using Theorem 2.3.10, Theorem 2.3.3 and Theorem 6.7.9, we immediately have

Theorem 6.7.10. *Let $\mathcal{S}$ be a GQ of order (s,t) with $s \neq 1 \neq t$, and suppose p is a point of $\mathcal{S}$.*

(1) *If $s = t$, then $\mathcal{S}$ is a TGQ with translation point p if and only if p is incident with three regular lines L_1, L_2, L_3 for which there are lines M_1, M_2, M_3 such that $L_i \sim M_i \not\mathrm{I} p$ and such that there are groups G_i of whorls about p which act transitively on the points of $M_i \setminus \{M_i \cap L_i\}$.*

(2) *If $s \neq t$, then $\mathcal{S}$ is a TGQ with translation point p if and only if p is incident with $t-s+3$ regular lines $L_0, L_1, \ldots, L_{t-s+2}$ for which there are lines $M_0, M_1, \ldots, M_{t-s+2}$ such that $L_i \sim M_i \not\mathrm{I} p$ and such that there are groups G_i of whorls about p which act transitively on the points of $M_i \setminus \{M_i \cap L_i\}$.*

In both cases, the translation group G is the group generated by the G_i's for all feasible i, and if $s \neq t$, then G is generated by $t-s+2$ arbitrary G_j's. ■

6.8 Recollection

Suppose $\mathcal{S}$ is an element of **II.2** of order (s,t), $s \neq 1 \neq t$, and let p be incident with $k+1$ axes of symmetry, $1 \leq k \leq t-1$.

(R1) By Theorem 2.3.3, we know that if $s = t$, then p is a translation point if $k > 1$, and hence $t > s$ if $k > 1$.

(R2) By Theorem 1.2.1, we have that $st(s+1) \equiv 0 \mod s+t$. Moreover, if $k > 1$, then by Theorem 6.4.1, $s|t$ and $\frac{t}{s}+1$ divides $(s+1)t$.

(R3) By Theorem 6.7.9, we have that $k \leq t-s+1$.

(R4) We have that if $t > s^2/2$, then $k \geq s+1$ implies that $k = t$, that is, $\mathcal{S}$ is a TGQ. Hence if $t > s^2/2$, we have that $k < s+1$. If for each point $x \not\sim p$ of $\mathcal{S}$, $|\{p,x\}^{\perp\perp}| = 2$, then the same remarks could be made.

(R5) Suppose that $k > 2$ (so $s \neq t$), and let $L_0, L_1, \ldots, L_k$ be the axes of symmetry of $\mathcal{S}$. Then there is no $i \in \{0,1,\ldots,k\}$ so that Property (T) is satisfied for (L_i, p) w.r.t. some other three distinct elements of $\{L_0, L_1, \ldots, L_k\}$ (cf. Theorem 6.2.2).

(R6) Assume that k and the L_j are as in (R5). Then $\mathcal{S}$ has no subGQ of order s which contains three distinct elements of $\{L_0, L_1, \ldots, L_k\}$ but not all.

(R7) By Theorem 2.3.15, p is not an elation point.

We do not have examples of **II.2** *at present.*

Conjecture. **II.2** *is empty.*

This seems like a reasonable conjecture to make, as the only known thick GQ's which are not EGQ's have order $(s-1, s+1)$ or $(s+1, s-1)$ for some s; see (R7) and recall the discussion of Chapter 5.

6.9 Remark on the Structure of Certain Groups which Act on Generalized Quadrangles

In this section, we will generalize a useful theorem of S. E. Payne and J. A. Thas concerning the structure of certain groups which act on generalized quadrangles, namely Theorem 2.1.3. For reasons of convenience, we recall that theorem here:

Theorem 6.9.1. *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t) with $s \leq t$ and $s > 1$, and let p be a point for which $\{p, x\}^{\perp\perp} = \{p, x\}$ for all $x \in P \setminus p^{\perp}$. Let G be a group of whorls about p.*

(1) *If $y \sim p$, $y \neq p$, and if θ is a nonidentity whorl about p and y, then all points fixed by θ lie on py and all lines fixed by θ meet py.*

(2) *If θ is a nonidentity whorl about p, then θ fixes at most one point of $P \setminus p^{\perp}$.*

(3) *If G is generated by elations about p, then G is a group of elations, i.e. the set of elations about p is a group.*

(4) *If G acts transitively on $P \setminus p^{\perp}$ and $|G| > s^2t$, then G is a Frobenius group on $P \setminus p^{\perp}$, so that the set of all elations about p is a normal subgroup of G of order s^2t acting regularly on $P \setminus p^{\perp}$, i.e. $\mathcal{S}^{(p)}$ is an EGQ with some normal subgroup of G as elation group.*

(5) *If G is transitive on $P \setminus p^{\perp}$ and G is generated by elations about p, then $(\mathcal{S}^{(p)}, G)$ is an EGQ.* ■

These observations appear to be crucial tools in many basic theorems for EGQ's and TGQ's, especially in recognition theorems for these quadrangles, see Chapter 8 of FGQ. The result of this section contributes (modestly) to that idea.

6.9.1 Definition of the closure

In view of Theorem 6.7.7, we have found the following notion to be useful in various situations, see, e.g., [141].

Let $\mathcal{S} = (P, B, I)$ be a generalized quadrangle of order (s, t), $s \neq 1 \neq t$, and let $L_0, L_1, \ldots, L_l$ be distinct axes of symmetry of $\mathcal{S}$ which are incident with the point p, $l \geq 2$. Suppose G is the group which is generated by the symmetries about $L_0, L_1, \ldots, L_l$. Let G_* be an arbitrary G-orbit in $P \setminus p^{\perp}$, and define the point-line geometry $\mathcal{S}(G_*) = (P', B', I')$ as follows:

- POINTS are of three types:
  (1) the point p;
  (2) the points of G_*;
  (3) all points different from p which are incident with lines of Type (a).
- LINES are of two types:
  (a) the lines L' incident with p which have the property that if $L'' \not\sim p$ is a line of $\mathcal{S}$ which is not skew to G_* and if $L'' \sim L'$, then L'' is incident with s points of G_*;
  (b) the lines of $\mathcal{S}$ which contain at least one point of G_* and which are concurrent with a line of Type (a).
- INCIDENCE is the natural one.

Then following [141], we call $\mathcal{S}(G_*) = (P', B', I')$ the *closure* of G_*, see also Section 6.10.

6.9.2 The structure of certain groups which act on generalized quadrangles

We are ready to prove

Theorem 6.9.2. *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t \neq 1$ and $s \neq t$, and let p be a point which is incident with $l+1$ axes of symmetry $L_0, L_1, \ldots, L_l$, $l \geq t/s$. Suppose that G is the group generated by the symmetries about $L_0, L_1, \ldots, L_l$, and suppose that $|G| \geq s^2t$. Assume that each nontrivial element of G has at most two fixed points in $P \setminus p^\perp$. Then the following properties hold.*

(1) *The group G acts transitively on $P \setminus p^\perp$ and hence s and t have the same parity.*

(2) *If no nontrivial element of G has fixed points in $P \setminus p^\perp$, then $(\mathcal{S}^{(p)}, G)$ is a TGQ with base-group G and base-point p.*

(3) *If every non-identical element of G has at most one fixed point in $P \setminus p^\perp$, then $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

(4) *The group G contains at least s^2t elations with center p, and when there are exactly s^2t such elations, then $\mathcal{S}^{(p)}$ is a TGQ for some base-group $H \leq G$.*

Proof. (1) First, we recall Burnside's classical theorem. Let (X, G) be a finite permutation group, and suppose k is the number of G-orbits in X. Then we have the following:

$$k|G| = \sum_{g \in G} f(g),$$

where $f(g)$ is the number of fixed points of g in X. Applying this theorem to our situation, we obtain

$$k|G| = s^2t + \sum_{g \neq 1} f(g),$$

and hence $k|G| < s^2t + |G|2$. Since $|G| \geq s^2t$, we then have that $(k-3)|G| < 0$. The last inequality implies that $k \in \{1, 2\}$, so there are at most two G-orbits in $P \setminus p^\perp$.

Suppose now that G does not act transitively on $P \setminus p^\perp$; this means that there are exactly two orbits in that permutation group, say G_1 and G_2. By Theorem 6.7.7 there exists an $m \in \mathbb{N}$, $s \leq m \leq t$, so that $|G_1| = s^2m$. Since there are only two G-orbits, it follows that $|G_2| = s^2(t-m)$. Assume that L is a line of $\mathcal{S}$ intersecting G_1 in n points, $0 < n < s$. Let MIp be so that $M \sim L$. Then M is not an axis of symmetry since $n < s$. It is also clear that every line $M' \sim M$ of $\mathcal{S}$ which intersects G_1 in at least one point, does intersect G_1 in exactly n points. Counting the lines of L^G, we then obtain that

$$\frac{s^2m}{n} \leq st \quad \Rightarrow \quad sm \leq nt. \tag{6.3}$$

Every line of L^G (and in particular the line L) intersects G_2 in precisely $(s-n)$ points, hence

$$s(t-m) \leq (s-n)t \quad \Rightarrow \quad nt \leq sm, \tag{6.4}$$

and thus $sm = nt$. So, every line of $\mathcal{S}$ intersects the G-orbit G_1, respectively G_2, in 0, s or n, respectively s, 0 or $s-n$, points.

Now suppose $L' \not\sim p$ is a line of $\mathcal{S}$ which hits G_1 in s points, and suppose that $\mathcal{S}(G_1)$, respectively $\mathcal{S}(G_2)$, is the closure of G_1, respectively G_2. Suppose k_1+1, respectively k_2+1, is the number of lines in $\mathcal{S}(G_1)$, respectively $\mathcal{S}(G_2)$, through p. Then $k_i \geq t/s$, $i = 1, 2$. Now count the points of G_1, respectively G_2, to obtain

$$|G_1| = s + (m-1)s + sk_1(s-1) + (t-k_1)s(n-1), \tag{6.5}$$

and

$$|G_2| = s + (t-m-1)s + sk_2(s-1) + (t-k_2)s(s-n-1). \tag{6.6}$$

If we then add Equalities (6.5) and (6.6) side by side, we obtain

$$2k_2s + k_1ns = s^2k_1 + snk_2 + st,$$

a contradiction using the facts that $t > s$ (as $\mathcal{S}$ has regular lines and $s \neq t$) and $t \leq s^2$. So there is no line of $\mathcal{S}$ intersecting G_1 in n points with $0 < n < s$. Hence $\mathcal{S}(G_1)$ is a substructure of $\mathcal{S}$ satisfying

(i) each line has $s+1$ points;

(ii) if two points of G_1 are collinear in $\mathcal{S}$, then they are also collinear in $\mathcal{S}(G_1)$.

By Theorem 1.3.1, this means that $\mathcal{S}(G_1)$ is a subGQ of $\mathcal{S}$ of order (s, t'), $t' \geq t/s$. But then it is clear that there must be some line intersecting G_1 in 1 point, a contradiction. So, there is only one G-orbit, and G acts transitively on $P \setminus p^\perp$.

(2) If no nontrivial element of G has fixed points in $P \setminus p^\perp$, then G acts regularly on $P \setminus p^\perp$, and $(\mathcal{S}^{(p)}, G)$ is an EGQ. Since G has at least two full groups of symmetries of order s about lines incident with p, this part of the theorem follows by Theorem 2.2.2 and Theorem 2.3.1.

(3) If every non-identical element of G has at most one fixed point in $P \setminus p^\perp$, and if there is at least one nontrivial element which has a fixed point in $P \setminus p^\perp$, then the transitive permutation group $(P \setminus p^\perp, G)$ has the following properties:

- G acts transitively, but not regularly on $P \setminus p^\perp$;
- there is no nontrivial element with more than one fixed point in $P \setminus p^\perp$.

Hence $(P \setminus p^\perp, G)$ is a Frobenius group, and the Frobenius kernel N acts regularly on $P \setminus p^\perp$. However, the definition of Frobenius kernel implies that N contains the symmetries about $L_0, L_1, \ldots, L_l$, and since G is generated by these symmetries, we have that $G = N$, contradiction. Hence G acts regularly on $P \setminus p^\perp$, and Part (2) yields the result.

(4) Let x_i be the number of elements of G which have exactly i fixed points in $P \setminus p^\perp$, $i \in \{0, 1, 2\}$. Then $|G| = 1 + x_0 + x_1 + x_2$. Applying the identity of Burnside and using the transitivity of the permutation group $(P \setminus p^\perp, G)$, we have that $|G| = s^2t + x_1 + x_2 2$, and so

$$s^2t = 1 + x_0 - x_2. \tag{6.7}$$

Since the identity is an elation (by definition), G contains at least s^2t elations with center p. If G would contain exactly s^2t elations with center p, then $x_2 = 0$ because of Equality (6.7). Part (3) now yields the result. ■

Remark 6.9.3. (i) For the case $s = t$, the theorem is trivially true; recall Theorem 1.2.2 and Theorem 2.3.3.

(ii) The application of Burnside's Theorem in the proof of the previous theorem turned out to be the key observation in handling Problems (1) and (2) of Section 2.1 in several general situations — see S. E. Payne and K. Thas [95], and K. Thas and H. Van Maldeghem [158].

6.10 Semi Quadrangles

It is not hard to verify that if a TGQ $(\mathcal{S}^{(x)}, G) = (P, B, I)$ of order (s, t), $s \neq 1 \neq t$, has a subGQ $\mathcal{S}'$ of order s which contains the translation point x (note that $\mathcal{S}'$ is then also a TGQ with translation group some subgroup of G), then we have the following fundamental property:

(Clo) *G has a subgroup of order s^3 which is generated by the symmetries about some three distinct lines through x, so that, for some G-orbit G_* in $P \setminus x^\perp$, $\mathcal{S}'$ precisely is the closure of G_*, that is, $\mathcal{S}' = \mathcal{S}(G_*)$.*

Thus, the closure of a general G_*-orbit (in the context of Section 6.9.1) seems the right geometry to study if no subGQ's are available (as was already illustrated a number of times). For an arbitrary subgroup G' of order s^3 of the translation group G which is generated by the symmetries about three distinct lines through x, one notes the following essential properties for the closure $\mathcal{S}(G_*)$, some of them which were already noted, for instance, in Theorem 6.7.7 (G_* being an arbitrary G'-orbit in $P \setminus x^{\perp}$):

(i) each line of $\mathcal{S}(G_*)$ is incident with $s+1$ points of $\mathcal{S}(G_*)$;

(ii) each point of $\mathcal{S}(G_*)$ is incident with at least three lines of $\mathcal{S}(G_*)$;

(iii) there are no triangles as subgeometry;

(iv) $\mathcal{S}(G_*)$ contains an ordinary quadrangle and pentagon as subgeometry.

Of course, if one takes the Properties (i)–(ii)–(iii)–(iv) to be the axioms of a point-line geometry Γ, then, as now Γ is not necessarily a subgeometry of a GQ (with $s+1$ points on a line), too many examples arise in order to have defined an 'interesting' axiomatic geometry. Note, however, that if the following axiom also is satisfied:

(SQ3) *For any two non-collinear points there is at least one point which is collinear with both*;

then the assumption that Γ is a subgeometry (in the usual sense) of a thick GQ $\mathcal{S}$ with $s+1$ points on a line, would imply that Γ is a thick subGQ of $\mathcal{S}$ of order (s, t') (see Theorem A.2.1 of Appendix A for a formal proof of this assertion). Moreover, the point-line geometries satisfying (i)–(ii)–(iii)–(iv)–(SQ3) which have a *linear representation* in $\mathbf{PG}(n, q)$ (for some q and $n \geq 2$), see Appendix A, are equivalent objects as *complete* $(k+1)$*-caps* of $\mathbf{PG}(n, q)$ if $q > 2$, see Theorem A.6.3.

We define a *semi quadrangle* as an incidence structure satisfying the Axioms (i)-(ii)-(iii)-(iv)-(SQ3); this definition will be formalized in Appendix A, and the above observations of this section, but also of Theorem 6.7.7 and of Section 6.9.1, motivate us to systematically study them in detail in that appendix.

Appendix: A New Short Proof of a Theorem of S. E. Payne and J. A. Thas

We start the appendix with recalling

Theorem 6.A.1. *Suppose $\mathcal{S} = (P, B, I)$ is a thick GQ of order s, and let p be a point of P incident with three distinct axes of symmetry L_1, L_2, L_3. Then every line through p is an axis of symmetry, and so $\mathcal{S}$ is a TGQ.*

The proof of this theorem which is mentioned in Chapter 11 of FGQ (Theorem 11.3.5, p. 246) is not easy, and uses a coordinatization method for GQ's of order s, and the theory of planar ternary rings. We will give a new geometrical proof without the use of coordinatization (but which uses a result of X. Chen and D. Frohardt, though).

In the following we define G as the group generated by all symmetries about L_1, L_2 and L_3. Furthermore, G_i will be the full group of symmetries about the axis L_i.

Proof of the Theorem. By Theorem 6.2.1, G is a group of order s^3 of elations with center p, thus, $(\mathcal{S}^{(p)}, G)$ is an EGQ. The theorem now follows from Theorem 2.3.1. ■

Chapter 7

Symmetry-Class $\geq$ III: Span-Symmetric Generalized Quadrangles

If a (thick) generalized quadrangle $\mathcal{S}$ has two non-concurrent axes of symmetry, then $\mathcal{S}$ is called a 'span-symmetric generalized quadrangle'. *If a GQ is as such, then as it is not contained in Symmetry-Classes* **I** *and* **II**, *we say that its symmetry-class is 'at least'* **III**.

In this chapter, we will prove the twenty-year-old conjecture that every span-symmetric generalized quadrangle of order s, $s \neq 1$, is classical, i.e. isomorphic to the generalized quadrangle $\mathcal{Q}(4, s)$ which arises from a nonsingular parabolic quadric in $\mathbf{PG}(4, s)$.

Historical Remark. It was conjectured in 1980 by S. E. Payne that a span-symmetric generalized quadrangle of order $s > 1$ is isomorphic to $\mathcal{Q}(4, s)$. There was a "proof" of this theorem as early as in 1981 by S. E. Payne which appeared in [75], but later on, it was noticed by the author himself that there was a mistake in that proof. The paper was very valuable however, since the author introduced there the '4-gonal bases', see Section 7.2, and proved for instance Theorem 7.2.3 and Theorem 7.2.4, below.
There was also an unpublished proof by W. M. Kantor from around that time. Recently, he has written down that proof, see [56] (which appeared in the same volume of *Advances in Geometry* as [145]).

The proof of the classification of span-symmetric GQ's of order (s, t), $1 < s \leq t < s^2$, is based on W. M. Kantor [56] and K. Thas [145], except for the sharply 2-transitive case with $s \neq t$; the combinatorial part of that proof is new, the other part is taken from [56]. Theorem 7.12.1 was obtained for the odd case in

K. Thas [147]; here, we present a general approach. Section 7.13 is based on the addendum of K. Thas [147]; Section 7.14 is taken from K. Thas [143]. Finally, Section 7.15 is a part of K. Thas [142].

7.1 First Main Results

In this part of the chapter, we will prove the following main result.

Theorem 7.1.1. *Let $\mathcal{S}$ be a span-symmetric generalized quadrangle of order s, where $s \neq 1$. Then $\mathcal{S}$ is classical, i.e. isomorphic to $\mathcal{Q}(4, s)$.*

This leads to the complete classification of groups which have a 4-*gonal basis* (as defined in Section 7.2).

Theorem 7.1.2. *A finite group is isomorphic to* $\mathbf{SL}(2, s)$ *for some s if and only if it has a 4-gonal basis.*

7.2 Span-Symmetric Generalized Quadrangles

Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, and suppose L and M are distinct non-concurrent axes of symmetry; then it is easy to see by transitivity that every line of $\{L, M\}^{\perp\perp}$ is an axis of symmetry, and $\mathcal{S}$ is called a *span-symmetric generalized quadrangle (SPGQ)* with *base-span* $\{L, M\}^{\perp\perp}$. Note that $|\{L, M\}^{\perp\perp}| = s + 1$, as L and M are regular lines.

Let $\mathcal{S}$ be a span-symmetric generalized quadrangle of order (s, t), $s, t \neq 1$, with base-span $\{L, M\}^{\perp\perp}$. Throughout this chapter, we will continuously use the following notation.
First of all, the base-span will always be denoted by $\mathcal{L}$. The group which is generated by all the symmetries about the lines of $\mathcal{L}$ is G, and sometimes we will call this group the *base-group*. This group clearly acts 2-transitively on the lines of $\mathcal{L}$, and fixes every line of $\mathcal{L}^{\perp}$. The set of all the points which are on lines of $\{L, M\}^{\perp\perp}$ is denoted by Ω (of course, Ω is also the set of points on the lines of $\{L, M\}^{\perp}$; we have that $|\{L, M\}^{\perp}| = |\{L, M\}^{\perp\perp}| = s + 1$). We will refer to $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$, with I' being the restriction of I to $(\Omega \times (\mathcal{L} \cup \mathcal{L}^{\perp})) \cup ((\mathcal{L} \cup \mathcal{L}^{\perp}) \times \Omega)$, as being *the base-grid*.

The substructure of fixed elements of an element of the base-group is given by the following.

Theorem 7.2.1 (FGQ, 10.7.1). *Let $\mathcal{S}$ be an SPGQ of order (s, t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$ and base-group G. If $\theta \neq \mathbf{1}$ is an element of G, then the substructure $\mathcal{S}_\theta = (P_\theta, B_\theta, I_\theta)$ of elements fixed by θ must be given by one of the following.*

(i) *$P_\theta = \emptyset$ and B_θ is a set of mutually non-concurrent lines containing $\mathcal{L}^{\perp}$.*

(ii) *There is a line $L \in \mathcal{L}$ for which P_θ is the set of points incident with L, and $M \sim L$ for each $M \in B_\theta$ ($\mathcal{L}^\perp \subseteq B_\theta$).*

(iii) *B_θ consists of $\mathcal{L}^\perp$ together with a subset B' of $\mathcal{L}$; P_θ consists of those points incident with lines of B'.*

(iv) *$\mathcal{S}_\theta$ is a subGQ of order (s, t') with $s \leq t' < t$. This forces $t' = s$ and $t = s^2$.* ■

We will also need the next result.

Theorem 7.2.2 (FGQ, 10.7.2). *If $\mathcal{S}$ is an SPGQ of order (s, t), $s, t \neq 1$ and $t < s^2$, with base-group G, then G acts regularly on the set of $(s+1)s(t-1)$ points of $\mathcal{S}$ which are not on any line of $\mathcal{L}$.* ■

In the second part of the chapter, we will come to an analogue of this result for the general case.

Let $\mathcal{S}$ be an SPGQ of order $s \neq 1$ with base-span $\mathcal{L}$, and put $\mathcal{L} = \{U_0, U_1, \ldots, U_s\}$. The group of symmetries about U_i is denoted by G_i, $i = 0, 1, \ldots, s$, throughout this chapter. Then one notes the following properties (see [75] and 10.7.3 of FGQ):

(1) the groups $G_0, G_1, \ldots, G_s$ form a complete conjugacy class in G, and are all of order s, $s \geq 2$;

(2) $G_i \cap N_G(G_j) = \{\mathbf{1}\}$ for $i \neq j$;

(3) $G_i G_j \cap G_k = \{\mathbf{1}\}$ for i, j, k distinct, and

(4) $|G| = s^3 - s$.

We say that G is a group with a 4-*gonal basis* $\mathfrak{B} = \{G_0, G_1, \ldots, G_s\}$ if these four conditions are satisfied.

It is possible to recover the SPGQ $\mathcal{S}$ of order s, $s > 1$, from the base-group G starting from 4-gonal bases, in the following way, see [75, 91].
Suppose G is a group of order $s^3 - s$ with a 4-gonal basis $\mathfrak{B} = \{G_0, G_1, \ldots, G_s\}$, and let $G_i^* = N_G(G_i)$ for $i = 0, 1, \ldots, s$. Define a point-line incidence structure $\mathcal{S}_\mathfrak{B} = (P_\mathfrak{B}, B_\mathfrak{B}, I_\mathfrak{B})$ as follows.

- $P_\mathfrak{B}$ consists of two kinds of POINTS.
  - (a) Elements of G.
  - (b) Right cosets of the G_i^*'s.
- $B_\mathfrak{B}$ consists of three kinds of LINES.
  - (i) Right cosets of G_i, $0 \leq i \leq s$.
  - (ii) Sets $M_i = \{G_i^* g \parallel g \in G\}$, $0 \leq i \leq s$.
  - (iii) Sets $L_i = \{G_j^* g \parallel G_i^* \cap G_j^* g = \emptyset, 0 \leq j \leq s, j \neq i\} \cup \{G_i^*\}$, $0 \leq i \leq s$.

- INCIDENCE. $I_{\mathfrak{B}}$ is the natural incidence: a line $G_i g$ of Type (i) is incident with the s points of Type (a) contained in it, together with that point $G_i^* g$ of Type (b) containing it. The lines of Types (ii) and (iii) are already described as sets of those points with which they are to be incident.

Then $\mathcal{S}_{\mathfrak{B}} = (P_{\mathfrak{B}}, B_{\mathfrak{B}}, I_{\mathfrak{B}})$ is a GQ of order s which is span-symmetric for the base-span $\{L_0, L_1\}^{\perp\perp}$ [75, 91]. Also, if $\mathcal{S}$ is an SPGQ of order s, $s \neq 1$, with base-span $\mathcal{L}$ and base-group G, and where $\mathfrak{B}$ is the corresponding 4-gonal basis, then $\mathcal{S} \cong \mathcal{S}_{\mathfrak{B}}$ [75, 91]. We thus have the following interesting theorem.

Theorem 7.2.3 (FGQ, 10.7.8). *An SPGQ of order $s \neq 1$ with given base-span $\mathcal{L}$ is canonically equivalent to a group G of order $s^3 - s$ with a 4-gonal basis $\mathfrak{B}$.* ■

It is also important to recall the following.

Theorem 7.2.4 (FGQ, 10.7.9). *Let $\mathcal{S}$ be an SPGQ of order $s \neq 1$, with base-span $\mathcal{L}$. Then every line of $\mathcal{L}^\perp$ is an axis of symmetry.* ■

This theorem thus yields the fact that for any two distinct lines U and V of $\mathcal{L}^\perp$, the GQ is also an SPGQ with base-span $\{U, V\}^{\perp\perp}$. The corresponding base-group will be denoted by $G^\perp$. It should be emphasized that this property only holds for SPGQ's of order s, $s > 1$ (as will be shown in the present work).

7.3 Split BN-Pairs of Rank 1

Recent developments in the theory of SPGQ's have shown that the notion of *(finite) split BN-pair of rank* 1 is particularly useful for various aims, as well as the theory of *perfect central extensions of (perfect) groups*. Let us first recall that a group with a *split BN-pair of rank* 1, or sometimes just a *split BN-pair of rank* 1, is a (not necessarily faithful) permutation group (X, H) which satisfies the following properties:

(BN1) H acts 2-transitively on X, $|X| > 2$;

(BN2) for every $x \in X$ there holds that the stabilizer of x in H has a normal subgroup H^x which acts regularly on $X \setminus \{x\}$.

Note. In [162], J. Tits has introduced the notion of *Moufang set*, which is essentially the same object as a split BN-pair of rank 1.

The elements of X are the *points* of the split BN-pair of rank 1, and for any x, the group H^x will be called a *root group*. An element of the group which is generated by all the root groups is a *transvection*, and the group H is the *transvection group*. If X is a finite set, then the split BN-pair of rank 1 also is called *finite* (note that, if the permutation group (X, H) is faithful, then H is also finite if X is finite).

The following theorem by E. Shult [107] and C. Hering, W. M. Kantor and G. M. Seitz [40] classifies all finite groups with a split BN-pair of rank 1 without invoking the classification of finite simple groups.

Theorem 7.3.1 ([107]; [40]). *Suppose (X, H) is a finite group with a split BN-pair of rank 1, and suppose $|X| = s + 1$, with $s \in \mathbb{N}_0$. If H is generated by the root groups of the split BN-pair, then H is always one of the following list (up to isomorphism):*

(i) *a sharply 2-transitive group on X;*

(ii) $\mathbf{PSL}(2, s)$*;*

(iii) *the Ree group $\mathbf{R}(\sqrt[3]{s})$ (also sometimes denoted by ${}^{\mathbf{2}}\mathbf{G_2}(\sqrt[3]{s})$), with $\sqrt[3]{s}$ an odd power of 3;*

(iv) *the Suzuki group $\mathbf{Sz}(\sqrt{s})$ (sometimes denoted by ${}^{\mathbf{2}}\mathbf{B_2}(\sqrt{s})$), with $\sqrt{s}$ an odd power of 2;*

(v) *the unitary group $\mathbf{PSU}(3, \sqrt[3]{s^2})$,*

each with its natural action of degree $s + 1$. ■

Every root group has order s. In the first case, (X, H) is a 2-transitive Frobenius group, and then $s + 1$ is the power of a prime (see, e.g., [31]); in all of the other cases, s is the power of a prime. Further, we have that $|\mathbf{PSL}(2, s)| = (s+1)s(s-1)$ or $(s + 1)s(s - 1)/2$, according as s is even or odd, and the group acts (sharply) 3-transitively on X if and only if s is even; in the other cases, we have that $|\mathbf{R}(\sqrt[3]{s})| = (s+1)s(\sqrt[3]{s}-1)$, $|\mathbf{Sz}(\sqrt{s})| = (s+1)s(\sqrt{s}-1)$, and $|\mathbf{PSU}(3, \sqrt[3]{s^2})| = \frac{(s+1)s(\sqrt[3]{s^2}-1)}{gcd(3,\sqrt[3]{s}+1)}$. For references on the orders of these groups, see [31], [43, p. 420-421] or Appendix B. A good reference for geometrical definitions of the groups of (ii)–(iii)–(iv)–(v) is the monograph [164] of H. Van Maldeghem, see also the lecture notes [140] by J. A. Thas, K. Thas and H. Van Maldeghem.

The 'natural action' in the Cases (ii)–(iii)–(iv)–(v) is the action of the group by conjugation on its Sylow p-subgroups, where s is a power of p. (Note that, as each of these groups admits a split BN-pair of rank 1, the root groups are *precisely* the Sylow p-subgroups.) This natural action can also be seen in a pure geometrical sense — see [164].

If $q > 3$, $\mathbf{PSL}(2, q)$ is a simple group. In the case $q = 2$, respectively $q = 3$, $\mathbf{PSL}(2, 2)$, respectively $\mathbf{PSL}(2, 3)$, is a (sharply) 2-transitive Frobenius group in its natural action on the projective line over $\mathbf{GF}(2)$, respectively $\mathbf{GF}(3)$.

The group $\mathbf{Sz}(q)$ is a simple group for $q \neq 2$. If $q = 2$, then $|\mathbf{Sz}(2)| = 20$, so $\mathbf{Sz}(2)$ is a (sharply) 2-transitive Frobenius group. Hence $\mathbf{Sz}(2)$ has a regular normal subgroup of size 5.

We have that $\mathbf{R}(q)$ is simple for $q \neq 3$. If $q = 3$, then $\mathbf{R}(q) \cong \mathbf{P\Gamma L}(2, 8)$, and $\mathbf{PSL}(2, 8)$ is a normal subgroup of $\mathbf{R}(3)$.

Remark 7.3.2. The root groups of $\mathbf{PSL}(2, s)$ and of the sharply 2-transitive groups are the only ones to be abelian (note that $\mathbf{Sz}(2)$ also has this property; this group acts sharply 2-transitively, however).

7.4 Remarks about 4-Gonal Bases

Now suppose G is a group of order $s^3 - s$, $s > 1$, where s is a power of the prime p, and suppose that G has a 4-gonal basis $\mathfrak{B} = \{G_0, G_1, \ldots, G_s\}$. Since the groups G_i all have order s, all these groups are Sylow p-subgroups in G. Since $\mathfrak{B}$ is a complete conjugacy class, this means that every Sylow p-subgroup of G is contained in $\mathfrak{B}$, and hence G has *exactly* $s+1$ Sylow p-subgroups. Hence we have proved the following easy but important theorem.

Theorem 7.4.1. *Suppose G is a group of order $s^3 - s$, $s > 1$, with s a prime power. Then G can have at most one 4-gonal basis. In particular, if G has a 4-gonal basis, then it is unique.* ■

As a corollary we obtain

Theorem 7.4.2. *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $1 < s \leq t < s^2$. Then $\mathcal{S}$ is isomorphic to the classical GQ $\mathcal{Q}(4,s)$ if and only if the base-group is isomorphic to $\mathbf{SL}(2,s)$.*

Proof. Suppose that the base-group G is isomorphic to $\mathbf{SL}(2,s)$; then s is the power of a prime, and hence by Theorem 7.4.1, $\mathbf{SL}(2,s)$ has at most one 4-gonal basis. As $t < s^2$, G acts regularly on the points of $\mathcal{S}$ not in Ω, so $(s+1)s(t-1) = (s+1)s(s-1)$, and $s = t$. Now consider $\mathcal{Q}(4,s)$ and suppose L and M are non-concurrent lines of $\mathcal{Q}(4,s)$. Then L and M are axes of symmetry, and hence $\mathcal{Q}(4,s)$ is span-symmetric for the base-span $\{L,M\}^{\perp\perp}$. In this case, the base-group is well known to be isomorphic to $\mathbf{SL}(2,s)$, see e.g. [75] and Remark 7.4.3, which proves that $\mathbf{SL}(2,s)$ indeed *has* a 4-gonal basis, that is necessarily unique by Theorem 7.4.1. Hence, by Theorem 7.2.3, there is only one GQ (up to isomorphism) which can arise from $\mathbf{SL}(2,s)$ using 4-gonal bases in the usual sense, and this is $\mathcal{Q}(4,s)$. Whence $\mathcal{S} \cong \mathcal{Q}(4,s)$. ■

Remark 7.4.3. The fact that the base-group corresponding to an arbitrary base-span $\mathcal{L}$ of $\mathcal{Q}(4,s)$ is isomorphic to $\mathbf{SL}(2,s)$, will also be shown implicitly in this chapter; this will in fact be a direct corollary of the main theorem of this part of the chapter.

7.5 Perfect and Universal Central Extensions of Perfect Groups

The following notions and results are taken from [3].
Let G be a group. Then the *derived group* of G, denoted G', is $\langle [g,h] \parallel g,h \in G \rangle$, where $[g,h] = g^{-1}h^{-1}gh$. First recall that a *perfect group* is a group G which equals its derived group. Suppose G and H are groups. Then H is called a *central extension* of G if there is a surjective homomorphism

$$\phi : H \longrightarrow G$$

for which $ker(\phi) \leq Z(H)$ ($ker(\phi)$ is the kernel of the homomorphism ϕ, $Z(H)$ is the center of H). Sometimes the pair (H, ϕ) is also called a *central extension* of G. A central extension $(\overline{G}, \xi)$ of a group G is called *universal*, if for any other central extension (H, ξ') of G there exists a unique homomorphism

$$\psi : \overline{G} \longrightarrow H$$

such that the diagram defined by

$$\begin{tikzcd} & H \arrow[dr, "\xi'"] & \\ \overline{G} \arrow[ur, "\psi"] \arrow[rr, "\xi"] & & G \end{tikzcd}$$

commutes. If a group G has a universal central extension $\overline{G}$, then $\overline{G}$ is known to be unique, up to isomorphism [3, 33.1].

Theorem 7.5.1 ([3], 33.4 (p. 167)). *A group G has a universal central extension if and only if it is perfect. The universal central extension of a group is always perfect if it exists.* ■

Using the preceding remarks and Theorem 7.5.1, it is possible to prove the following result:

Theorem 7.5.2 ([3], 33.8(1) (p. 168)). *Suppose G is a perfect group, and suppose $\overline{G}$ is its universal central extension. Furthermore, let H be a perfect group which is a central extension of G. Then there exists a subgroup N of the center $Z(\overline{G})$ of $\overline{G}$, such that*

$$\overline{G}/N \cong H.$$

■

7.6 SPGQ's and Split BN-Pairs of Rank 1

Theorem 7.6.1. *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, t), $s, t \neq 1$, with base-span $\mathcal{L}$ and base-group G. Let N be the kernel of the action of G on $\mathcal{L}$. Then G/N either acts as a sharply 2-transitive group on $\mathcal{L}$, or is isomorphic (as a permutation group) to one of the following:*

(i) $\mathbf{PSL}(2, s)$;

(ii) *the Ree group* $\mathbf{R}(\sqrt[3]{s})$;

(iii) *the Suzuki group* $\mathbf{Sz}(\sqrt{s})$;

(iv) *the unitary group* $\mathbf{PSU}(3, \sqrt[3]{s^2})$,

each with its natural action of degree $s + 1$.

Proof. The group G (and hence also G/N) is doubly transitive on $\mathcal{L}$, and for every $L \in \mathcal{L}$ the full group of symmetries about L, which acts regularly on $\mathcal{L} \setminus \{L\}$, is a normal subgroup of the stabilizer of L in G. This means that $(\mathcal{L}, G/N)$ is a split BN-pair of rank 1. Theorem 7.3.1 provides the above list of possibilities for G/N, noting that G/N is generated by the normal subgroups mentioned above. ∎

For the rest of this section, we suppose that $s \leq t < s^2$.

Lemma 7.6.2. *We have that G is a perfect group if G/N does not act sharply 2-transitively on $\mathcal{L}$, and if $G/N \not\cong \mathbf{R}(3)$.*

Proof. Suppose G/N does not act sharply 2-transitively on $\mathcal{L}$. By Theorem 7.6.1, we have that G/N is isomorphic to one of the following: (a) $\mathbf{PSL}(2, s)$; (b) $\mathbf{R}(\sqrt[3]{s})$; (c) $\mathbf{Sz}(\sqrt{s})$; (d) $\mathbf{PSU}(3, \sqrt[3]{s^2})$. All these groups are simple and hence perfect groups, except when $G/N \cong \mathbf{R}(3)$. So if $G/N \cong \mathbf{R}(\sqrt[3]{s})$, we assume that $s \neq 27$. Assume that G is distinct from its derived group G'. Then since G/N is a perfect group, we have that

$$(G/N)' = G'N/N = G/N,$$

and hence $G'N = G$.

Suppose that $s = t$.

First suppose we are in Case (a). If s is even, then $|G| = |\mathbf{PSL}(2, s)|$, and thus $|N| = 1$. So in that case $G = G'$, a contradiction. If s is odd, then G' is a subgroup of G of index 2. It follows that G and G' have exactly the same Sylow p-subgroups (note that $G' \trianglelefteq G$), with s a power of the odd prime p, as s is the largest power of p which divides $|G|$ and $|G'|$. Since here G is generated by its Sylow p-subgroups (by the definition of the base-group G), we infer that $G = G'$, a contradiction. Hence G is perfect.

Now suppose we are in Case (b) or (c). Then $|N| = \frac{s-1}{s^n-1}$ with $n \in \{1/2, 1/3\}$, and hence $|N|$ and s are mutually coprime since $s-1$ and s are mutually coprime. Hence s is a divisor of $|G'|$, since $|G| = \frac{|G'| \times |N|}{|G' \cap N|}$. Thus G and G' have precisely the same Sylow p-subgroups, with s a power of the prime p. Since G is generated by its Sylow p-subgroups, we conclude that $G = G'$, a contradiction.

Finally, assume that we are in the last case. Then $|N| = \frac{gcd(3, \sqrt[3]{s}+1)(s-1)}{\sqrt[3]{s^2}-1}$, and thus it is clear that $|N|$ and s are mutually coprime. The same argument as before yields that $|G'| \equiv 0 \mod s$, and hence that $G = G'$, a contradiction. Consequently G is perfect.

Now suppose that $s < t < s^2$.

For this case, we use the internal structure of $G_{U_i}/N = G_i^*/N$, where $U_i \in \mathcal{L}$ is arbitrary but fixed, to show that G_i is a subgroup of G' (and hence to conclude

the theorem). In each of the groups we consider, $G_iN/N \cong G_i$ lies in $(G_i^*/N)'$. It follows easily that $G_i \leq (G_i^*)'$, so that $G_i \leq G'$, and the result follows. (In the proof of Lemma 7.10.2, we will work this out in detail for one example with a general argument.) ■

The proof for the case $s < t < s^2$ also works when $t = s^2$. Later on in this chapter, we will produce a different (more geometrical) argument for the latter case, except if $G/N \cong \mathbf{PSL}(2,s)$ and s is even; then, a similar method will be utilized.

Remark 7.6.3. For $s = 2$, the GQ is isomorphic to $\mathcal{Q}(4,2)$. In this case $G = G/N \cong S_3$ acts sharply 2-transitively on $\mathcal{L}$.

Lemma 7.6.4. *N is contained in the center of G.*

Proof. Clearly N is a normal subgroup of G. Let H be the full group of symmetries about an arbitrary line of $\mathcal{L}$. Then $N \cap H = \{\mathbf{1}\}$ and N and H normalize each other, thus they commute. As G is generated by the symmetries about the lines of $\mathcal{L}$, the lemma follows. ■

Lemma 7.6.5. *If $\mathcal{S}$ is an SPGQ of order (s,t), $1 < s \leq t < s^2$, with base-group G and base-span $\mathcal{L}$, then G/N acts either as $\mathbf{PSL}(2,s)$ or as a sharply 2-transitive group on the lines of $\mathcal{L}$.*

Proof. Assume by way of contradiction that G/N does not act as $\mathbf{PSL}(2,s)$ or a sharply 2-transitive group on the lines of $\mathcal{L}$. First of all, G is a perfect group except (possibly) when $G/N \cong \mathbf{R}(3)$, and since N is in the center of G, the group G is a perfect central extension of the group G/N if $G/N \not\cong \mathbf{R}(3)$. The perfect group G/N has a universal central extension $\overline{G/N}$, and $\overline{G/N}$ contains a central subgroup F such that $\overline{G/N}/F \cong G$. We now look at the possible cases.

If $G/N \cong \mathbf{Sz}(\sqrt{s})$, and if $s > 8^2$, then N must be trivial since in that case the Suzuki group has a trivial universal central extension (i.e. $\overline{G/N} \cong G/N$) by Appendix B, an impossibility since the orders of G and $\mathbf{Sz}(\sqrt{s})$ are not the same if $t \geq s > 8^2$. Suppose that $s = 8^2$. Then by Appendix B any perfect central extension H of $\mathbf{Sz}(8)$ satisfies $|H| = 2^k|\mathbf{Sz}(8)|$ for some $k \in \{0,1,2\}$. None of these cases occurs since $|G| \geq (64)^3 - 64 = 262080$ and since $|\mathbf{Sz}(8)| = 29120$.

If $G/N \cong \mathbf{R}(\sqrt[3]{s})$, $s > 27$, then we have exactly the same situation as in the preceding (general) case (i.e. the universal central extension of $\mathbf{R}(\sqrt[3]{s})$ is trivial), compare Appendix B, hence this case is excluded as well.
Suppose $G/N \cong \mathbf{R}(3) \cong \mathbf{P\Gamma L}(2,8)$. Then G/N contains a normal subgroup $S/N \cong \mathbf{PSL}(2,8)$ of index 3. For fixed $i \in \{0,1,\ldots,s\}$, note that $|G_i \cap S| = |(G_i \cap S)N/N| = 9$ (as S/N cannot contain G_iN/N; otherwise, as S/N is a normal subgroup of G/N, S/N would contain all the G_i's, and hence $S = G$, contradiction). Let H be the subgroup of G generated by the G-conjugates of $G_i \cap S$ — so

$$H = \langle g^{-1}(G_i \cap S)g \parallel g \in G\rangle.$$

Then $HN/N = S/N$ and $G_i \cap H = G_i \cap S$. Also,

$$G_i \cap H \cong (G_i \cap H)N/N \leq (H_{L_i}N/N)' \leq (HN/N)' = H'N/N,$$

so $G_i \cap H \leq H'$. Hence $H = H'$, and $H/Z(H) \cong \mathbf{PSL}(2,8)$, so that $Z(H) = \{\mathbf{1}\}$ and $H \cong \mathbf{PSL}(2,8)$. By definition, H is transitive on the G-conjugates of $G_i \cap H$, and hence also on the G-conjugates of G_i, so that HG_i contains all such conjugates, and hence $G = HG_i$. From

$$|G| = |H| \times \frac{|G_i|}{|G_i \cap H|} = |\mathbf{R}(3)|$$

now follows that $G = \mathbf{R}(3)$, producing the same contradiction as before.

Finally, assume that $G/N \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$. The universal central extension of the group $\mathbf{PSU}(3, \sqrt[3]{s^2})$ is known to be $\mathbf{SU}(3, \sqrt[3]{s^2})$, see Appendix B, and also, we know that $|\mathbf{SU}(3, \sqrt[3]{s^2})| = gcd(3, \sqrt[3]{s}+1)|\mathbf{PSU}(3, \sqrt[3]{s^2})| = (s+1)s(\sqrt[3]{s^2}-1)$. This provides us with a contradiction as $t - 1 \geq s - 1 > \sqrt[3]{s^2} - 1$.

This proves the assertion. ∎

Lemma 7.6.6. *If G/N acts as $\mathbf{PSL}(2,s)$, then $G \cong \mathbf{SL}(2,s)$ and $\mathcal{S}$ is classical. Moreover, we have that $s = t$.*

Proof. The universal central extension of $\mathbf{PSL}(2,s)$ is $\mathbf{SL}(2,s)$, except in the cases $s = 4$ and $s = 9$, compare Appendix B, and in general $|\mathbf{SL}(2,s)| = gcd(2, s-1) \cdot |\mathbf{PSL}(2,s)| = |G|$. Hence if $s \neq 4, 9$, G is isomorphic to $\mathbf{SL}(2,s)$, and by Theorem 7.4.2, $\mathcal{S}$ is classical.
There is a unique GQ of order 4, namely $\mathcal{Q}(4,4)$, see Chapter 1 (Section 1.6), so $s = 4$ gives no problem; in this case, G is isomorphic to $\mathbf{SL}(2,4)$. Finally, suppose that $s = 9$. Then there is only one possible perfect central extension of $G/N \cong \mathbf{PSL}(2,9)$ with size at least $9^3 - 9 = 234$, namely $\mathbf{SL}(2,9)$, see Appendix B. Hence $G \cong \mathbf{SL}(2,9)$, and by Theorem 7.4.2, $\mathcal{S}$ is classical and isomorphic to $\mathcal{Q}(4,9)$. ∎

7.7 The Sharply 2-Transitive Case

Recall that if $\mathcal{S}$ is an SPGQ of order $s \neq 1$ with base-span $\mathcal{L}$, then every line of $\mathcal{L}^\perp$ is an axis of symmetry.

First suppose that G/N acts as a sharply 2-transitive group on the lines of $\mathcal{L}$ in the SPGQ $\mathcal{S}$ of order $s > 1$.

Since the lines of $\mathcal{L}^\perp$ are also axes of symmetry, we can assume that the base-group $G^\perp$ corresponding to these lines also acts as a sharply 2-transitive group on $\mathcal{L}^\perp$, because otherwise $G^\perp$ is isomorphic to $\mathbf{SL}(2,s)$, and then $\mathcal{S}$ is classical by Theorem 7.4.2. Hence G and $G^\perp$ contain (normal) central subgroups N and $N^\perp$

which act trivially on the points of Ω, both of order $s-1$ (where Ω is the set of points on the lines of the base-span). Note that G and $G^\perp$ act regularly on the points of $\mathcal{S}$ not in Ω by Theorem 7.2.2.

Let p be a point and L a line of a projective plane Π. Then Π is said to be (p, L)*-transitive* if the group of all collineations of Π with center p and axis L acts transitively on the points, distinct from p and not on L, of any line through p. The following theorem is a step in the Lenz-Barlotti classification of finite projective planes, see e.g. [27, 111]; it states that the Lenz-Barlotti Class **III.2** is empty.

Theorem 7.7.1 (J. C. D. S. Yaqub [169]). *Let Π be a finite projective plane, containing a non-incident point-line pair (x, L) for which Π is (x, L)-transitive, and assume that Π is (y, xy)-transitive for every point y on L. Then Π is Desarguesian.* ■

As every axis of symmetry L is a regular line, there is a projective plane Π_L canonically associated with L as in Theorem 4.1.1. Hence

Theorem 7.7.2. *Suppose that $\mathcal{S}$ is an SPGQ of order s, where $s \neq 1$, with base-group G and base-span $\mathcal{L}$. Also, let N be the kernel of the action of G on the lines of $\mathcal{L}$, and suppose that G/N acts as a sharply 2-transitive group on the lines of $\mathcal{L}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,2)$ or $\mathcal{Q}(4,3)$.*

Proof. Fix a line L of $\mathcal{L}$, and consider the projective plane Π_L^* of order s, which is the dual of Π_L. Then $\mathcal{L}^\perp$ is a point of Π_L^* which is not incident with L as a line of the plane. For convenience, denote this point by p. Now consider the action of N as a collineation group on Π_L^*. Clearly, this action is faithful (recall that N fixes Ω pointwise). Then, as $|N| = s-1$ and as N fixes L pointwise and p linewise, the plane Π_L^* is (p, L)-transitive. Now fix an arbitrary line U through p in Π_L^*; then U is a line of $\mathcal{L}^\perp$. If we interpret the group $G_U^\perp$ of all symmetries about U as a collineation group of Π_L^* (this is possible since $G_U^\perp$ fixes L), then $G_U^\perp$ fixes every line through the point $L \cap U$ of Π_L^*. Suppose r is an arbitrary point of Π_L^* on U and different from $L \cap U$. Then, in the GQ, r is of the form $\{U, U'\}^{\perp\perp}$, with U' some line of $L^\perp$ which does not meet U. It is clear that for any symmetry θ about U we have $(\{U, U'\}^{\perp\perp})^\theta = \{U, U'\}^{\perp\perp}$, and thus any element of $G_U^\perp$, as a collineation of Π_L^*, fixes every point on the line U. From the fact that $|G_U^\perp| = s$, and that distinct elements of $G_U^\perp$ induce distinct collineations of Π_L^*, it follows that Π_L^* is $(U \cap L, U)$-transitive. Hence by Theorem 7.7.1 the plane Π_L^* is Desarguesian.
Now consider the action of the groups $G_V^\perp$ on Π_L^*, with $V \in \mathcal{L}^\perp$ (and where the notation is obvious). Then $G_V^\perp$ fixes the line L and the point $V \cap L$ and acts regularly on the other points of L. The group $G^\perp = \langle G_V^\perp \parallel V \in \mathcal{L}^\perp \rangle$, as a collineation group of the plane, induces a sharply 2-transitive permutation group on the points of L by our hypothesis. But since the plane Π_L^* is Desarguesian, we also know that the groups $G_V^\perp$, as collineation groups of the plane, generate a $\mathbf{PSL}(2, s)$ on L. For $s > 3$, the fact that $\mathbf{PSL}(2, s)$ acts sharply 2-transitively on L, implies that $|\mathbf{PSL}(2, s)| = (s+1)s$, a contradiction.

Finally, suppose that $s = 2$, respectively $s = 3$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,2)$, respectively to $\mathcal{Q}(4,3)$ (cf. Section 1.6, and recall the fact that $\mathcal{S}$ has regular lines). ■

Now suppose that $s < t < s^2$.

There is a purely combinatorial way to get rid of the sharply 2-transitive case for $s < t < s^2$, where $t \neq s+2$ if s is even, as follows.

Let $s < t < s^2$, and suppose G acts sharply 2-transitively on $\mathcal{L}$. If $x \in \mathcal{S} \setminus \Omega$, then x^N is a set of $t-1$ points which are all collinear with each point of $x^\perp \cap \Omega = X$. Counting the number of points of $\mathcal{S}$, we obtain

$$(s+t) + (s^2+s) + (s+1)(t-1)(s-1) + \frac{(t-1)(t-s)s}{\overline{p}} \leq (s+1)(st+1),$$

where $\overline{p}$ is the average number of points of x^N which are collinear with a point z of $\mathcal{S} \setminus \Omega$ which is not incident with a line that contains a point of x^N and X, but which is collinear with at least one point of x^N. It follows that $t - s \leq \overline{p}$. Clearly, $\overline{p} \leq t - s$ (a line which joins z and a point of x^N does not hit Γ), so $\overline{p} = t - s$. As $(\overline{p} - 1)s = s(t-s-1) = t-2$ (by considering an arbitrary line M hitting x^N but not Γ, and counting the points of x^N not on M which are collinear with M), we have that $s = t - 2$. Note that this also follows from Theorem 1.4.5. Since there are nontrivial symmetries about lines of $\mathcal{S}$, Section 5.2 implies that s is even (note that s plays a different role there!).

We now produce a separate proof for this case (which also works for the general case $s < t < s^2$).

So suppose again that $s < t < s^2$. First of all, since G/N is a sharply 2-transitive group on $\mathcal{L}$, we have that $s+1 = p^h$ for some prime h, and there is an elementary abelian normal subgroup K/N of order p^h. Since $N \leq Z(K)$, $K/Z(K)$ is nilpotent (as a p-group), so K is nilpotent (see p. 29 of [3]). It follows that K has a unique Sylow p-subgroup P, cf. p. 23 of [35]. (For the special case $t = s+2$, there also is the following alternative, and more elementary, argument: $|K| = (t-1)p^h = p^{2h}$, so that K is a p-group, thus coinciding with P.)

Since P/N is an abelian group, we have that

$$(P/N)' = \{\mathbf{1}\} = P'N/N,$$

from which follows that $P' \leq N$.

Now recall the following application of Maschke's Theorem, cf. p. 66, 177 of [35]:

Theorem 7.7.3. *Let H' be an automorphism group of the abelian r-group H (r a prime), with gcd$(|H'|, r) = 1$. Then*

$$H = C_H(H') \times [H, H'].$$

■

Here, $C_H(H')$ is the subgroup of H which is elementwise fixed by each element of H', and $[H,H'] = \langle h^{-1}h^{h'} \parallel h \in H, h' \in H'\rangle$. Note that both $C_H(H')$ and $[H,H']$ are H'-invariant subgroups of H.

Fix some $i \in \{0,1,\ldots,s\}$. If one puts $H' = G_i$ and $H = P/P'$, where the action of G_i on P/P' is natural, then

$$P/P' = (N/P') \times [P/P', G_i].$$

So there is some $B/P' \le P/P'$ for which

$$P/P' = (N/P') \times (B/P'),$$

and B/P' is G_i-invariant. Hence B is normalized by G_i, so that B is normalized by $G = \langle P, G_i\rangle$. We can conclude that $BG_i = G_iB$ contains all the G-conjugates of G_i, implying that $BG_i = G$. It follows that $P' = N$. Now let $h \in P\setminus N$, and define $A = A(h) = \{[h,h'] \parallel h' \in P\}$ (which is a subset of $P' = N$). As $[hN,h'] = [h,h']$ for any $h' \in P$, the 2-transitivity of G/N implies that $A(h)^G = [P,P] = A(h)$ (note that $g^{-1}[h,h']g = [h^g,(h')^g]$). As $[h,h'N] = [h,h']$ for any h' and $[h,N] = \{\mathbf{1}\}$, it follows that

$$|A| \le |P/N| - 1 = s,$$

so that

$$|N| = |P'| = |A| = t-1 \le s,$$

implying that $t = s+1$. This contradicts the standard divisibility condition for the parameters of a GQ. ■

7.8 Classification of SPGQ's of Order (s,t), $1<s\le t<s^2$

Theorem 7.8.1. *Let $\mathcal{S}$ be an SPGQ of order (s,t), where $1 < s \le t < s^2$. Then $s = t$ and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$.*

Proof. Adopt the notation $G, N, G/N, \mathcal{L}$, etc. from above. By Lemma 7.6.5, G/N either acts as a sharply 2-transitive group on $\mathcal{L}$, or as $\mathbf{PSL}(2,s)$. If G/N acts as $\mathbf{PSL}(2,s)$, then $G \cong \mathbf{SL}(2,s)$, $s=t$ and $\mathcal{S}$ is classical by Lemma 7.6.6. If G/N acts sharply 2-transitively on $\mathcal{L}$, then by Section 7.7, $\mathcal{S} \cong \mathcal{Q}(4,2)$ or $\mathcal{Q}(4,3)$. Whence the result. ■

This leads to the complete classification of groups having a 4-gonal basis:

Theorem 7.8.2. *A finite group is isomorphic to $\mathbf{SL}(2,s)$ for some s if and only if it has a 4-gonal basis.* ■

It also follows that the order of an SPGQ is essentially known:

Theorem 7.8.3. *Let $\mathcal{S}$ be an SPGQ of order (s,t), $1 < s,t$. Then $t \in \{s,s^2\}$.* ■

7.9 On the Size of the Base-Group (General Case)

The following lemma is extremely easy but essential.

Lemma 7.9.1. *Let $\mathcal{S}$ be an SPGQ of order (s,t), $s \neq 1 \neq t$, with base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$ and base-group G, and let U_i, G_j, etc. be as before. If p is a point which is not an element of Ω, and U is a line through p which meets Ω in a certain point qIU_k of Γ, then every point on U which is different from q is a point of the G-orbit which contains p.*

Proof. Immediate by the action of G_k. ■

The proof of the next lemma will be used several times in the rest of this work. It was first obtained in K. Thas [147].

Lemma 7.9.2. *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$, with base-grid Γ and base-group G. Then G has size at least $s^3 - s$.*

Proof. If $s = t$, then we already know that G has order $s^3 - s$, so suppose $s \neq t$ (and then $t = s^2$ by Theorem 7.8.3).
Set $\mathcal{L} = \{U_0, U_1, \ldots, U_s\}$ and suppose G_i is the group of symmetries about U_i for all feasible i. Suppose p is a point of $\mathcal{S}$ not in Ω, and consider the following $s+1$ lines $M_i := proj_p U_i$. If Λ is the G-orbit which contains p, then by Lemma 7.9.1 every point of M_i not in Ω is also a point of Λ. Now consider an arbitrary point $q \neq p$ on M_0 which is not on a line of $\mathcal{L}$. Then again every point of $proj_q U_i$ not in Ω is a point of Λ. Hence we have the following inequality:

$$|\Lambda| \geq 1 + (s+1)(s-1) + (s-1)^2 s, \tag{7.1}$$

from which it follows that $|\Lambda| \geq s^3 - s^2 + s$. Now fix a line U of $\mathcal{L}^{\perp}$. Every line of $\mathcal{S}$ which meets this line and which contains a point of Λ is completely contained in $\Lambda \cup \Omega$ by Lemma 7.9.1. Also, G acts transitively on the points of U. Suppose k is the number of lines through a (= every) point of U that are completely contained in $\Lambda \cup \Omega$ (as point sets). If we count in two ways the number of point-line pairs (u, M) for which $u \in \Lambda, M \sim U$ and uIM, then it follows that

$$\begin{aligned} k(s+1)s &= |\Lambda| \geq s(s^2 - s + 1) \\ &\implies k \geq \frac{s^2 - s + 1}{s+1} = s - 2 + \frac{3}{s+1}, \end{aligned}$$

and hence, since $k \in \mathbb{N}$, we have that $k \geq s - 1$. Thus $|\Lambda| \geq s^3 - s$ and so also $|G|$. ■

Remark 7.9.3. Another easy way to obtain the previous lemma is by counting the number of $(s+1) \times (s+1)$-grids which are completely contained in $\Lambda \cup \Omega$ (as point sets), and that contain a fixed line of $\mathcal{L}$. (In such a way, one can obtain more general results; see, e.g., K. Thas [152, 155].)

7.10 SPGQ's of Order (s, s^2), $s > 1$: Determination of the Base-Group

In K. Thas [147], we proved that if $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$, where $s \neq t$ and s is odd, then $\mathcal{S}$ contains $s + 1$ subquadrangles, all isomorphic to the classical GQ $\mathcal{Q}(4, s)$, which mutually intersect in the base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$. Also, the base-group G acts semiregularly on the points of $\mathcal{S} \setminus \Omega$ and $G \cong \mathbf{SL}(2, s)$. Note that $|G| = |\mathbf{SL}(2, s)| = s^3 - s$. In this section, it is our goal to obtain the same result for s even (as a corollary of a proof which works in any characteristic). This result will be crucial for the study of several cases to be obtained in Theorem 9.1.3.

Remark 7.10.1. Already since the very beginning of the study of SPGQ's (late 1970's), it was thought of that SPGQ's of order (s, s^2), $s > 1$, always have ('many') classical subGQ's of order s, all passing through the base-grid. In [147], we solved that conjecture for the odd case. In this section, the problem is completely solved for the general case. We emphasize that this is a very strong result, which will be illustrated many times in the rest of this work.

Suppose $\mathcal{S}$ is an SPGQ of order (s, s^2), $s > 1$, with base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$ and base-group G ($\mathcal{L}$ is like before). For convenience, we will suppose that $s > 3$; if $s = 2$, then $\mathcal{S} \cong \mathcal{Q}(5, 2)$ by Section 1.6. In that case, there are $s+1 = 3$ subGQ's of order 2, isomorphic to $\mathcal{Q}(4, 2)$ and mutually intersecting in the base-grid. The base-group stabilizes each of these subGQ's (as it is generated by symmetries), and acts semiregularly on the points of $\mathcal{S}$ not in Ω; whence $G \cong \mathbf{SL}(2, 2)$. Exactly the same properties hold for the case $s = 3$; see Section 1.6.

First suppose that G does not act semiregularly on $\mathcal{S} \setminus \Omega$. Suppose that $\theta \neq \mathbf{1}$ is an element of G which fixes a point q of $\mathcal{S} \setminus \Omega$.
Then by Theorem 7.2.1, the fixed element structure of θ is a subGQ $\mathcal{S}_{\theta}$ of order s. It is clear that $\mathcal{S}_{\theta}$ is also span-symmetric with respect to the same base-span. Hence $G_{\theta} := G/N_{\theta}$, with N_{θ} the kernel of the action of G on $\mathcal{S}_{\theta}$ — we can speak of 'an action' since G stabilizes $\mathcal{S}_{\theta}$ — has order $s^3 - s$; G_{θ} is exactly the base-group corresponding to $\mathcal{L}$ seen as a base-span of $\mathcal{S}_{\theta}$. Also, by Theorem 7.8.1, we have that $\mathcal{S}_{\theta} \cong \mathcal{Q}(4, s)$, and

$$G_{\theta} \cong \mathbf{SL}(2, s).$$

Next, let x be an arbitrary point of $\mathcal{S} \setminus \mathcal{S}_{\theta}$, and consider the set of points $V = x^{\perp} \cap \mathcal{S}_{\theta}$. Note that $|V| = t + 1 = s^2 + 1$ because $\mathcal{S}_{\theta}$ is a GQ of order s (every line hits $\mathcal{S}_{\theta}$). Then x cannot be fixed by θ — as otherwise $x \in \mathcal{S}_{\theta}$ — and $\{x, x^{\theta}\} \subseteq V^{\perp}$. Since $\mathcal{S}$ has order (s, s^2), we know by Theorem 1.4.5 that

$$|\{x, x^{\theta}\}^{\perp\perp}| = 2,$$

and thus, as N_{θ} acts semiregularly on the points of $\mathcal{S}$ outside $\mathcal{S}_{\theta}$, N_{θ} has size 2 (N_{θ} is not trivial). So, N_{θ} is a normal subgroup of G of order 2 and N_{θ} is thus

contained in the center of G.
It is important to note that $|G| = 2(s^3 - s)$. We also note that $(\mathbf{SL}(2,s))' = \mathbf{SL}(2,s)$ since $s > 3$; see D. E. Taylor [112, p. 22].

Observe that

$$(\mathbf{SL}(2,s))' = \mathbf{SL}(2,s) \cong (G/N_\theta)' = G/N_\theta = G'N_\theta/N_\theta,$$

so that $G'N_\theta = G$.

Lemma 7.10.2. *If G does not act semiregularly on $\mathcal{S} \setminus \Omega$, then G is a perfect group.*

Proof. Suppose $G \neq G'$. First suppose that s is odd.
As $G'N_\theta = G$, G and G' have exactly the same Sylow p-subgroups, where $s = p^h$, $h \in \mathbb{N}$. The group G is generated by its subgroups G_i, and since these are precisely the Sylow p-subgroups, we have that $G = G'$, contradiction. Hence G is perfect.
We now use the internal structure of $\mathbf{SL}(2,s) \cong \mathbf{PSL}(2,s)$ to obtain the even case. (The argument for this case will be similar as the argument at the end of the proof of Lemma 7.6.2.)
Consider a nontrivial θ in G_0, and an arbitrary nontrivial element ϕ of $(G_{U_0})_{U_i}$ (note that ϕ is nontrivial in its action on $\mathcal{L}$), $i \neq 0$. Note also that ϕ fixes no line of $\mathcal{L} \setminus \{U_0, U_i\}$; the action of $\mathbf{PSL}(2,s)$ on $\mathcal{L}$ is sharply 3-transitive. Then $[\theta, \phi^{-1}] = \theta^{-1}\phi\theta\phi^{-1}$ is a nontrivial symmetry about U_0, and it follows easily that $\langle [\theta, \phi^{-1}] \parallel \phi \in (G_{U_0})_{U_i} \rangle = G_0$. Whence $G_0 \leq G'_{U_0} \leq G'$. So $G = \langle G_i \parallel i \in \{0, 1, \ldots, s\} \rangle \leq G'$, and G is a perfect group. ∎

Now observe

Lemma 7.10.3. *G acts semiregularly on $\mathcal{S} \setminus \Omega$.*

Proof. As before, we can suppose that $s > 3$. Suppose by way of contradiction that G does not act semiregularly on the points of $\mathcal{S} \setminus \Omega$. We then know that $G/N_\theta \cong \mathbf{SL}(2,s)$, where N_θ is a central subgroup of order 2. The group G equals its derived group by Lemma 7.10.2, and has size $2(s^3 - s)$. The universal central extension of $\mathbf{SL}(2,s)$ coincides with $\mathbf{SL}(2,s)$ if $s \neq 4, 9$, and this contradicts the fact that $|G| = 2(s^3 - s)$. Hence G *does* act semiregularly on the points of $\mathcal{S} \setminus \Omega$ if $s \neq 4, 9$.

Now put $s = 4$. Then $\mathcal{S}$ is a GQ of order $(4, 16)$ which contains a subGQ isomorphic to $\mathcal{Q}(4,4)$, hence by J. A. Thas [117], $\mathcal{S} \cong \mathcal{Q}(5,4)$. The lemma then follows for this case.

Finally, suppose that $s = 9$. Then if $\overline{\mathbf{PSL}(2,9)}$ is the universal central extension of $\mathbf{PSL}(2,9)$, four possibilities can occur for the perfect central extension G of $\mathbf{PSL}(2,9)$:

(1) $G = \mathbf{PSL}(2,9)$;

(2) $G \cong \overline{\mathbf{PSL}(2,9)}/\mathbb{Z}_2$;

(3) $G \cong \mathbf{SL}(2,9) \cong \overline{\mathbf{PSL}(2,9)}/\mathbb{Z}_3$;

(4) $G \cong \overline{\mathbf{PSL}(2,9)}$.

As G is a perfect central extension of $\mathbf{SL}(2,9)$, the fact that $|G| = 4|\mathbf{PSL}(2,9)| = 2|\mathbf{SL}(2,9)|$ produces the contradiction.

Hence G acts semiregularly on $\mathcal{S} \setminus \Omega$. ■

Suppose that $\mathcal{S}$ is an SPGQ of order (s, s^2), $s \neq 1$, with base-span $\mathcal{L} = \{U, V\}^{\perp\perp}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I^*)$. Furthermore, let N be the kernel of the action of G on $\mathcal{L}$. As before, we assume w.l.o.g. that $s > 3$.
Throughout the proof of the next lemma, we assume that G acts semiregularly on $\mathcal{S} \setminus \Omega$.

Lemma 7.10.4. *G/N cannot act as a sharply 2-transitive group on the lines of $\mathcal{L}$ if $s > 3$.*

Proof. Suppose G/N acts as a sharply 2-transitive group on the lines of $\mathcal{L}$. Then $|G/N| = (s+1)s$. Hence, since $|G| \geq s^3 - s$ by Lemma 7.9.2, we have that $|N| \geq s - 1$. Let q be an arbitrary point of $\mathcal{S} \setminus \Omega$, and define V as $V := q^{\perp} \cap \Omega$ (so $|V| = s+1$). Then since G acts semiregularly on the points of $\mathcal{S}\setminus\Omega$, $|N| = |q^N|$, and $q^N \subseteq V^{\perp}$. As $\mathcal{S}$ is a thick SPGQ of order (s,t) with $s \neq t$, Theorem 7.8.3 implies that $t = s^2$, hence every triad of points has exactly $s+1$ centers by Theorem 1.4.1. So, we immediately have that $|N| = |q^N| \leq s + 1$.
Now suppose that $|N| \neq s - 1$, so that $|N| \in \{s, s+1\}$. First suppose $|N| = s$. Then the order of G is $s^2(s+1)$, and by the semiregularity condition this must be a divisor of $|\mathcal{S} \setminus \Omega| = (s+1)(s^3 - s)$, clearly a contradiction. Hence $|N| = s + 1$.

Assume that s is odd.
Since G/N is supposed to be a sharply 2-transitive group in its action on $\mathcal{L}$, $|G| = (s+1)^2 s$. Suppose that Λ is a G-orbit in $\mathcal{S} \setminus \Omega$; as G acts semiregularly on $\mathcal{S} \setminus \Omega$, $|\Lambda| = |G|$. Consider an arbitrary point p in Λ. Then every point of $\mathbf{X} = p^N$ is collinear with every point of $\mathbf{Y} = p^{\perp} \cap \Omega$, and we denote the set of points which are on a line of the form uv with $u \in \mathbf{X}$ and $v \in \mathbf{Y}$ by $\mathbf{XY}$. It is clear that $\mathbf{XY} \setminus \mathbf{Y}$ is completely contained in Λ, and that the order of this set is $(s+1)s^2$. Now take a point q of Λ outside $\mathbf{XY}$. The points of $\mathbf{X}$ and $\mathbf{Y}$ are the points of a dual grid with parameters $s+1, s+1$, and hence, if x, y and z are arbitrary distinct points of $\mathbf{Y}$ (or of $\mathbf{X}$), the triad $\{x, y, z\}$ is 3-regular. Put $q^N = \{q = q^0, q^1, q^2, \ldots, q^s\}$. If q^i is an arbitrary point of q^N, then $|\mathbf{Y} \cap (q^i)^{\perp}| =: k_{q_i} \leq 2$. One notes that $k_{q_i} = k_{q_j} =: k$ for some constant k, and that $\mathbf{Y} \cap (q^i)^{\perp} = \mathbf{Y} \cap (q^j)^{\perp}$, for all i and j, by the action of N. If W is an arbitrary line through q which intersects Ω, then W does not contain a point of $\mathbf{X}$ since this would imply that q is not outside $\mathbf{XY}$. Recalling Section 1.4 (and recalling the fact that s is odd), we count the number of points which are collinear with a point of q^N and which are contained in Λ, together with the points of $\mathbf{XY} \cap \Lambda$. One notes that every point of q^N is

collinear with every point of $q^\perp \cap \Omega$, and also that q^N is skew to $\mathbf{XY}$. We obtain the following:

$$\begin{aligned}|\Lambda| &= (s+1)^2 s\\ &\geq (s+1)s^2+s+1+k(s+1)(s-1) \ +(s+1-k)(s+1)(s-3),\end{aligned}$$

with $k \in \{0,1,2\}$, implying that $s<4$, a contradiction. Hence this case is excluded.

Now suppose s is even.
From the fact that $|N| = s+1$, follows that $\mathbf{X}$ and $\mathbf{Y}$ determine a 3-regular triad, and hence a subGQ $\mathcal{S}'$ of order s (by Section 1.4). Thus $|\mathbf{XY}| = |\mathcal{S}'|$. Take a point q of Λ outside $\mathbf{XY}$. Then every line which is incident with q and which intersects Ω has only that intersection point in common with $\mathcal{S}'$. Counting the number of points which are collinear with a point of q^N and which are contained in Λ, together with the points of $\mathbf{XY}\cap\Lambda$, we get the following inequality:

$$|\Lambda| = (s+1)^2 s \geq (s+1)s^2+s+1+(s+1)^2(s-1),$$

and thus $s \geq s^3$, a contradiction if $s \geq 2$.

Finally, suppose $|N| = s-1$ (so $|\Lambda| = (s+1)s(s-1)$). Let $W \in \mathcal{L}^\perp$, and let Λ be as before. By the semiregularity of G on the points of $\mathcal{S}\setminus\Omega$, the fact that $|G| = (s+1)s(s-1)$ and the fact that G acts transitively on the points incident with W, we have that any point on W is incident with exactly $s-1$ lines of $\mathcal{S}$ which are completely contained in Λ except for the point on W which is in Ω, and every point of Λ is incident with a line which meets W (recall that G is generated by groups of symmetries). Now define the following incidence structure $\mathcal{S}' = (P', B', I')$.

- Lines. The elements of B' are the lines of $\mathcal{S}'$ and they are essentially of two types:
  (1) the lines of $\mathcal{L}\cup\mathcal{L}^\perp$;
  (2) the lines of $\mathcal{S}$ which contain a point of Λ and a point of Ω.
- Points. The elements of P' are the points of the incidence structure and they are just the points of $\Omega\cup\Lambda$.
- Incidence. Incidence I' is the induced incidence.

Then any point of $\mathcal{S}'$ is incident with $s+1$ lines of $\mathcal{S}'$ and any line of $\mathcal{S}'$ is incident with $s+1$ points of the structure, and there are exactly $(s+1)(s^2+1)$ points and equally many lines. Whence one can easily conclude that $\mathcal{S}'$ is a generalized quadrangle of order s (since it is an induced subgeometry of a GQ, it cannot contain triangles). Clearly, $\mathcal{S}'$ is span-symmetric, so that $\mathcal{S}'$ is isomorphic to $\mathcal{Q}(4,s)$ by Theorem 7.8.1. Theorem 7.8.1 also asserts that $G \cong \mathbf{SL}(2,s)$, contradiction, as G/N then only acts sharply 2-transitively on $\mathcal{L}$ for $s \in \{2,3\}$. ■

The construction technique of subGQ's at the end of the preceding proof will inspire us to construct subGQ's in general SPGQ's (of order (s, s^2)) in Section 7.11.

We now have

Lemma 7.10.5. *G is a perfect group.*

Proof. We repeat a similar argument as in the case $s = t$. Let N be the kernel of the action of G on $\mathcal{L}$. As G/N does not act as a sharply 2-transitive group on $\mathcal{L}$, we have that G/N is isomorphic to one of the following list: (a) $\mathbf{PSL}(2, s)$, (b) $\mathbf{R}(\sqrt[3]{s})$, (c) $\mathbf{Sz}(\sqrt{s})$, or (d) $\mathbf{PSU}(3, \sqrt[3]{s^2})$, each with its natural permutation representation of degree $s + 1$. All these groups are perfect groups, except when $s = 27$ and $G/N \cong \mathbf{R}(3)$. We may exclude the latter case by a similar argument as that in the case $s = t$. Hence, since G/N is a perfect group,

$$(G/N)' = (G/N) = G'N/N \cong G'/(G' \cap N),$$

and thus $G'N = G$.
Since G acts semiregularly on $\mathcal{S} \setminus \Omega$, the natural number

$$|G| = |G/N| \times |N| = \frac{(s^n - 1)(s + 1)s|N|}{r},$$

with $r \in \{1, 2, gcd(3, \sqrt[3]{s} + 1)\}$ and $n \in \{1, 2/3, 1/2, 1/3\}$, is a divisor of $|\mathcal{S} \setminus \Omega| = (s + 1)(s^3 - s)$, where $r = 2$ if and only if s is odd and $G/N \cong \mathbf{PSL}(2, s)$ and where $r = gcd(3, \sqrt[3]{s}+1)$ if and only if $G/N \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$. Hence, we have that

$$r(s^2 - 1)/(s^n - 1) \equiv 0 \mod |N|. \tag{7.2}$$

First suppose that s is odd and that $G/N \not\cong \mathbf{PSL}(2, s)$.
If $r = gcd(3, \sqrt[3]{s} + 1)$ and $G/N \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$, then s and $|N|$ have a nontrivial common divisor if and only if $r = 3$ and if 3 is a divisor of s, clearly in contradiction with $3 = gcd(3, \sqrt[3]{s}+1)$. It follows now immediately from (7.2) that $|N|$ and s are coprime if s is odd and $G/N \not\cong \mathbf{PSL}(2, s)$. Hence with $s = p^h$ for the odd prime p and $h \in \mathbb{N}_0$, s is the largest power of p which divides $|G|$. Thus the full groups of symmetries about the lines of $\mathcal{L}$ are exactly the Sylow p-subgroups of G. Now suppose $G \neq G'$.
We know that $|N|$ and s are coprime, so since

$$|G'| = (|G| \times |G' \cap N|)/|N|,$$

there follows that $|G'| \equiv 0 \mod s$, whence G and $G' \trianglelefteq G$ have the same Sylow p-subgroups. But $G' \leq G$ and G is generated by its Sylow p-subgroups, so $G = G'$, a contradiction. Hence G is perfect.

Next, suppose that s is odd and that $G/N \cong \mathbf{PSL}(2, s)$.
Then we know that $|G| = |G/N| \times |N| = |N| \times |\mathbf{PSL}(2, s)| = |N| \times \frac{(s^3-s)}{2}$ is a divisor of $(s + 1)(s^3 - s)$, and again s and $|N|$ are mutually coprime. In the same

way as before, it follows now that G is a perfect group. The even case is handled in the same way. ■

Lemma 7.10.6. *N is in the center of G.*

Proof. Clearly N is a normal subgroup of G. Let H be the full group of symmetries about an arbitrary line of $\mathcal{L}$. Then $N \cap H = \{\mathbf{1}\}$ and N and H normalize each other, thus they commute. As G is generated by the symmetries about the lines of $\mathcal{L}$, the statement follows. ■

Lemma 7.10.7. *G/N is isomorphic to $\mathbf{PSL}(2, s)$ if $s > 3$.*

Proof. Assume by way of contradiction that G/N does not act as $\mathbf{PSL}(2, s)$. First of all, G is a perfect group, and since N is in the center of G, the group G is a perfect central extension of the group G/N. The perfect group G/N has a universal central extension $\overline{G/N}$, and $\overline{G/N}$ contains a central subgroup F such that $\overline{G/N}/F \cong G$. We now look at the possible cases.

If $G/N \cong \mathbf{Sz}(\sqrt{s})$, and if $s > 8^2$, then N must be trivial since in that case the Suzuki group has a trivial universal central extension, see Appendix B, an impossibility since the orders of G and $\mathbf{Sz}(\sqrt{s})$ are not the same if $s > 8^2$. Suppose that $s = 8^2$. Then by Appendix B any perfect central extension H of $\mathbf{Sz}(8)$ satisfies $|H| = 2^k|\mathbf{Sz}(8)|$ for some $k \in \{0, 1, 2\}$. None of these cases occurs since $|G| = (64)^3 - 64 = 262080$ and since $|\mathbf{Sz}(8)| = 29120$.

If $G/N \cong \mathbf{R}(\sqrt[3]{s})$, $s > 27$, then we have the same situation as in the preceding (general) case (the universal central extension of $\mathbf{R}(\sqrt[3]{s})$ is trivial), compare Appendix B, hence this case is excluded as well. (Recall that the case $G/N \cong \mathbf{R}(3)$ cannot occur.)

Now assume that $G/N \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$. The universal central extension of $\mathbf{PSU}(3, \sqrt[3]{s^2})$ is $\mathbf{SU}(3, \sqrt[3]{s^2})$, see Appendix B, and $|\mathbf{SU}(3, \sqrt[3]{s^2})| = gcd(3, \sqrt[3]{s} + 1)|\mathbf{PSU}(3, \sqrt[3]{s^2})| = (s+1)s(\sqrt[3]{s^2} - 1)$. This provides us with a contradiction since $s > 1$, hence $s - 1 > \sqrt[3]{s^2} - 1$. ■

7.11 Construction of Subquadrangles

Suppose $\mathcal{S}$ is an SPGQ of order (s, s^2), $s \neq 1$, with base-span $\mathcal{L} = \{U, V\}^{\perp\perp}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I^*)$. Then G acts semiregularly on the points of $\mathcal{S} \setminus \Omega$. Assume that G has order $(s + 1)s(s - 1)$. Let Λ be an arbitrary G-orbit in $\mathcal{S} \setminus \Omega$, and fix a line W of $\mathcal{L}^{\perp}$. By the semiregularity of G on the points of $\mathcal{S} \setminus \Omega$, the fact that $|G| = (s+1)s(s-1)$ and the fact that G acts transitively on the points incident with W, we have that any point on W is incident with exactly $s - 1$ lines of $\mathcal{S}$ which are completely contained in Λ except for the point on W which is in Ω, and every point of Λ is incident with a line which meets W (recall that G is generated by groups of symmetries). Now define the following incidence structure $\mathcal{S}' = \mathcal{S}'(\Lambda) = (P', B', I')$.

- Lines. The elements of B' are the lines of $\mathcal{S}'$ and they are essentially of two types:
  (1) the lines of $\mathcal{L} \cup \mathcal{L}^\perp$;
  (2) the lines of $\mathcal{S}$ which contain a point of Λ and a point of Ω.
- Points. The elements of P' are the points of the incidence structure and they are just the points of $\Omega \cup \Lambda$.
- Incidence. Incidence I' is the induced incidence.

Then by the proof of Lemma 7.10.4, $\mathcal{S}'$ is an SPGQ of order s. By Theorem 7.8.1, the GQ $\mathcal{S}'$ is isomorphic to the GQ $\mathcal{Q}(4, s)$, as $\mathcal{S}'$ is clearly span-symmetric for the base-span $\mathcal{L}$.

Theorem 7.11.1. *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle with base-span $\mathcal{L}$ and of order (s, s^2), $s \neq 1$. Assume that G has size $s^3 - s$. Then there exist $s + 1$ subquadrangles of order s which are all isomorphic to $\mathcal{Q}(4, s)$, so that they mutually intersect in the base-grid Γ.*

Proof. From each G-orbit in $\mathcal{S} \setminus \Omega$ there arises a subGQ of order s which is isomorphic to $\mathcal{Q}(4, s)$. There are exactly $s + 1$ such distinct G-orbits. ■

7.12 The Main Structure Theorem for SPGQ's of Order (s, s^2)

We are ready to prove the main result for SPGQ's of order (s, s^2).

Theorem 7.12.1. *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, s^2), $s \neq 1$. Then $\mathcal{S}$ contains $s+1$ subquadrangles, all isomorphic to the classical GQ $\mathcal{Q}(4, s)$, which mutually intersect in the base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$. Also, the base-group G acts semiregularly on $\mathcal{S} \setminus \Omega$, $|G| = s^3 - s$ and $G \cong \mathbf{SL}(2, s)$.*

Proof. The semiregularity of G on $\mathcal{S} \setminus \Omega$ implies that G is a perfect group, and also that $G/N \cong \mathbf{PSL}(2, s)$ if N is the kernel of the action of G on $\mathcal{L}$. Moreover, N is contained in the center of G. Thus, G is a perfect central extension of $\mathbf{PSL}(2, s)$ of size at least $s^3 - s$, and this leads to the fact that $G \cong \mathbf{SL}(2, s)$ and that $|G| = s^3 - s$.
The result follows from Section 7.11. ■

We will frequently use this observation without further notice.

7.13 Digression: Spreads in SPGQ's

Observe the following easy theorem.

Theorem 7.13.1. *An SPGQ of order (s,t), $s \neq 1 \neq t$, contains spreads if and only if $s = t$ with s even or if $t = s^2$.*

Proof. If $s = t$, then $\mathcal{S} \cong \mathcal{Q}(4,s)$ by Theorem 7.8.1, and then by 3.4.1 of FGQ the statement follows. If $s \neq t$, then $t = s^2$ by Theorem 7.8.3. Suppose Λ is an arbitrary G orbit in $\mathcal{S} \setminus \Omega$, where Ω is the set of points incident with the lines of the base-span $\mathcal{L}$. Then by Theorem 7.12.1, $\Lambda \cup \Omega$ is the set of points of a subGQ $\mathcal{S}' \cong \mathcal{Q}(4,s)$, and there are $s+1$ such subGQ's which arise in this way. Let L be an arbitrary line of $\mathcal{S}$ which is not contained in $\mathcal{S}'$, and which does not intersect Ω. Then L intersects $\mathcal{S}'$ in exactly one point. Since $|\Lambda| = s^3 - s$, the action of G on the points of $\mathcal{S} \setminus \Omega$ is semiregular, and any of the $s+1$ subGQ's of order s which contain Ω, is fixed by G. Now consider the line set L^G. Then $|L^G| = s^3 - s$ and no two distinct lines of L^G intersect (since the action of G on Λ is regular and since there are $s+1$ such G-orbits in $\mathcal{S} \setminus \Omega$). Now put $\mathbf{T} = \mathcal{L} \cup L^G$ and $\mathbf{T}' = \mathcal{L}^{\perp} \cup L^G$. Then $\mathbf{T}$ and $\mathbf{T}'$ clearly are spreads of $\mathcal{S}$. ■

If $\mathbf{T} = \mathcal{L} \cup L^G$ is as in the previous proof, then we denote $\mathbf{T}$ also by $\mathbf{T}(\mathcal{L}, L)$.

7.14 The "Hole" in the Moufang Theorem

In 1978, S. E. Payne and J. A. Thas started the program to prove the Moufang Theorem of P. Fong and G. M. Seitz [32, 33], cf. Theorem A, for finite generalized quadrangles, without the use of 'deep group theory' (and hence geometrically more satisfactory), see Chapter 9 of FGQ. They came very close to obtaining a proof; their only obstacle was (essentially) the following problem:

PROBLEM. *Suppose that $\mathcal{S}$ is a thick GQ of order (s, s^2), all lines of which are axes of symmetry. Then $\mathcal{S} \cong \mathcal{Q}(5,s)$.*

W. M. Kantor gave a proof of this theorem in [54], where he used the classification of (finite) split BN-pairs of rank 1, and 4B,C of P. Fong and G. M. Seitz [32], but the proof is still not elementary in the sense of S. E. Payne and J. A. Thas.

Theorem 7.14.1 ([32, 33]; [54]). *If $\mathcal{S}$ is a GQ of order (s, s^2), $s > 1$, each line of which is an axis of symmetry, then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.* ■

We do not finish the aforementioned program here, but we give a proof of Theorem 7.14.1 without the use of any result of [32, 33]; we still have to invoke the classification of (finite) split BN-pairs of rank 1, however. As will be pointed out, we do not need that result 'intrinsically'; we will at some stage only need the order of some base-group (it is there that our geometrical proof fails). The proof given in the next section is taken from Section 8.4 of K. THAS, *Automorphisms and characterizations of finite generalized quadrangles* [143].

Sketch of the Proof of the Theorem

We have shown that given an SPGQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$ and $s \neq t$ (and hence $t = s^2$), with base-group G and base-grid Γ, we have $|G| \geq s^3 - s$. Moreover, if $|G| = s^3 - s$, then there are $s+1$ subGQ's of order s, all isomorphic to $\mathcal{Q}(4,s)$ and mutually intersecting in Γ. Although we used Theorem 7.8.1 to show that these subGQ's are classical, this result is not needed here since all lines of each such subGQ are regular, as each line of $\mathcal{S}$ is regular (cf. Theorem 1.5.2). To prove that $|G|$ equals $s^3 - s$ (so that we can apply Section 7.11), we need the work of E. Shult [107] and C. Hering, W. M. Kantor and G. M. Seitz [40] (cf. Theorem 7.3.1), see Section 7.12. Since $\mathcal{S}$ is an SPGQ for any two non-concurrent lines, there then easily follows by e.g. Theorem 1.5.5 that $\mathcal{S} \cong \mathcal{Q}(5,s)$. ∎

Remark 7.14.2. (i) Although we did not completely finish the program of S. E. Payne and J. A. Thas here, we will obtain very powerful generalizations of the latter result, see, e.g., the main result(s) of the next chapter, and Section 12.2.

(ii) Very recently, J. A. Thas [131] has shown that a finite thick TGQ each line of which is regular, is classical, except, possibly, when the kernel of the translation generalized quadrangle is $\mathbf{GF}(2)$. If the kernel is $\mathbf{GF}(2)$, then the generalized quadrangle is classical if there is another translation point (a result that was first obtained by the author of this work in [147], see Theorem 8.1.1).

7.15 Generalization

The following result is taken from [142]. If p is a point of a GQ $\mathcal{S} = (P,B,I)$ of order (s,t), $s,t \neq 1$, such that there is a group of whorls about p which acts transitively on $P \setminus p^\perp$, then we call p a *center of transitivity*.

Theorem 7.15.1 (K. Thas [142]). *Suppose $\mathcal{S} = (P,B,I)$ is a GQ of order (s,t), $s,t \neq 1$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$ if and only if $\mathcal{S}$ contains a center of transitivity p, a collineation θ of $\mathcal{S}$ for which $p^\theta \not\sim p$, and a regular line.*

Proof. If $\mathcal{S}$ is isomorphic to one of $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$, then any point of $\mathcal{S}$ is a translation point, and hence every point is a center of transitivity. Also, every line is regular.
Conversely, suppose that $\mathcal{S}$ contains a center of transitivity p with a group G_p of whorls about p acting transitively on $P \setminus p^\perp$, a collineation θ of $\mathcal{S}$ for which $p^\theta \not\sim p$, and a regular line. Then it is easy to see that every point of $\mathcal{S}$ is a center of transitivity, and that $Aut(\mathcal{S})$ acts transitively on the lines of $\mathcal{S}$. Whence every line of $\mathcal{S}$ is a regular line, and by Theorems 1.5.2 and 1.5.1, we have one of the following possibilities:

(1) $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4, s)$;

(2) $t > s$.

Now consider the last case. Since every line is regular and every point is a center of transitivity, we have by Theorem 2.3.11 that every line of $\mathcal{S}$ is an axis of symmetry. The theorem now follows from Theorem 7.14.1. ■

Chapter 8

Generalized Quadrangles with Distinct Translation Points

In Chapter 2 and Chapter 6 (and Appendix A), we have studied the generalized quadrangles with some concurrent axes of symmetry in detail. In Chapter 7, we completely classified the span-symmetric generalized quadrangles of order (s,t), $1 < s \leq t < s^2$, by proving that they are isomorphic to $\mathcal{Q}(4,s)$. As two of the main goals in this work are:

(i) to obtain a classification of generalized quadrangles with axes of symmetry;

(ii) to develop a theory for those generalized quadrangles with non-concurrent axes of symmetry, that is, the span-symmetric generalized quadrangles ((ii) being a step towards (i)),

we proceed with the study of span-symmetric generalized quadrangles of order (s,s^2).

Although it was conjectured in 1981 by S. E. Payne that every SPGQ of order (s,s^2), $s > 1$, is isomorphic to $\mathcal{Q}(5,s)$, see PROBLEM 26 of [76], such a similar result as in the case $t < s^2$ cannot hold.
For, let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3,q)$, q odd. Then the q planes π_t with equation

$$tX_0 - mt^\sigma X_1 + X_3 = 0,$$

where $t \in \mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$ and σ a given field automorphism of $\mathbf{GF}(q)$, define a Kantor semifield flock $\mathcal{F}$ of $\mathcal{K}$. In 2001, S. E. Payne noted to us that the dual Kantor flock generalized quadrangles are span-symmetric (see his paper [79], where this observation is rather hidden), and this infinite class of generalized quadrangles contains non-classical examples; $\mathcal{F}$ is not a linear flock (and then the GQ is non-classical) if and only if σ is not the identity.

Moreover, every non-classical dual Kantor flock GQ even contains a line L for which every line which meets L is an axis of symmetry[1]! Hence there is some line each point of which is a translation point. In order to initialize a theory for thick span-symmetric generalized quadrangles of order (s, s^2) with $s \neq 1$, we therefore will first study a converse of the observation of S. E. Payne:

PROBLEM. *Classify all thick generalized quadrangles with distinct translation points.*

Another goal of this chapter is to state elementary combinatorial and group theoretical conditions for a GQ $\mathcal{S}$ such that $\mathcal{S}$ essentially arises from a flock (see also J. A. Thas [121], [125], [127], [126], [129], Chapter 6 and Chapter 11).

Chapter 8 is based on the paper K. Thas [147].

Remark (On the Notion of Translation Point). Often in the existing literature, one speaks of 'the translation point' of a given non-classical TGQ $\mathcal{S}$. Since each SPGQ of order s, $s > 1$, is classical, each non-classical TGQ of order s, $s > 1$, has a unique, and whence well-defined, translation point. As noted in the beginning of this chapter, rather surprisingly there are non-classical examples of TGQ's of order (s, s^2), $s > 1$, with distinct translation points, so TGQ's for which the notion 'the translation point' is not well-defined! In Chapters 8 and 10, we will completely determine those TGQ's.

8.1 The Main Theorem

Theorem 8.1.1. *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s, t), $s \neq 1 \neq t$, with two distinct collinear translation points. Then we have the following possibilities:*

(i) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4, s)$;*

(ii) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5, s)$;*

(iii) *$t = s^2$, $s = q^n$ with q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, $q \geq 4n^2 - 8n + 2$ and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock;*

(iv) *$t = s^2$, $s = q^n$ with q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, $q < 4n^2 - 8n + 2$ and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

If a thick GQ $\mathcal{S}$ has two non-collinear translation points, then $\mathcal{S}$ is always of classical type, i.e. isomorphic to one of $\mathcal{Q}(4, s)$, $\mathcal{Q}(5, s)$.

[1] We refer the reader to Chapter 10 for a proof of this result, and generalizations.

Proof. Let $\mathcal{S}$ be as assumed. If $\mathcal{S}$ contains two non-collinear translation points, then it is clear that Theorem 7.15.1 can be applied to obtain that $\mathcal{S}$ is isomorphic to one of $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$.
Suppose that $\mathcal{S}$ contains two distinct collinear translation points u and v. Then for $s=t$, the statement follows immediately from Theorem 7.8.1. If $s \neq t$, then $t=s^2$ by Theorem 7.8.3. By transitivity, every point of $L := uv$ is a translation point, and hence every line of $L^{\perp}$ is an axis of symmetry. If we fix some base-span $\mathcal{L}$ for which $L \in \mathcal{L}^{\perp}$, then by Theorem 7.12.1 there are $s+1$ classical subGQ's of order s which mutually intersect precisely in the base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$ (with the usual notation). Fix an arbitrary translation point $(\infty) I L$, and consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ with base-point (∞). Then clearly $\mathcal{O}$ is good at its element π which corresponds to L. If s is even, the result follows by Theorem 2.5.4.
Now let s be odd. Then the translation dual $T(\mathcal{O}^*)$ of $T(\mathcal{O})$ satisfies Property (G) for the flag $((\infty)', L')$ by Theorem 2.5.1, where $(\infty)'$ corresponds to (∞), and L' to L. If we now apply Theorem 2.7.4, then we obtain that $T(\mathcal{O}^*)$ is the point-line dual of a flock generalized quadrangle $\mathcal{S}(\mathcal{F})$. The theorem now follows from Theorem 3.4.3 and the fact that the dual Kantor semifield flock quadrangles are isomorphic to their translation duals. ∎

Corollary 8.1.2. *A generalized quadrangle of order (s,t), $s \neq 1 \neq t$ and s even, has two distinct collinear translation points if and only if $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$.* ∎

Remark 8.1.3. An SPGQ with base-span $\{U,U'\}^{\perp\perp}$ defines a split BN-pair of rank 1, where the root groups are the the full groups of symmetries about the lines of $\{U,U'\}^{\perp\perp}$. Suppose $\mathcal{S}$ is a thick GQ with two distinct collinear translation points p and q, and put $pq = L$. Suppose M and N are arbitrary non-concurrent lines of $L^{\perp}$. Since $L \cap M$ and $L \cap N$ are translation points of the GQ, the groups of symmetries about M and N are abelian groups (the translation groups corresponding to $L \cap M$ and $L \cap N$ are abelian), and hence every full group of symmetries about a line of $\{N,M\}^{\perp\perp}$ is abelian. The only transvection groups of a finite split BN-pair of rank 1 with abelian root groups of order s are isomorphic to $\mathbf{PSL}(2,s)$, or a sharply 2-transitive group, cf. Remark 7.3.2. It is thus possible to alternatively prove Theorem 8.1.1 without the full strength of Theorem 7.8.1 (only using a part of its proof).

8.2 Generalizations of Theorem 8.1.1

8.2.1 A useful lemma

In this section it is our aim to generalize Theorem 8.1.1 in various ways. We start with a lemma.

Lemma 8.2.1. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, and suppose L, p and q are so that L is a regular line, pIL is a point which is incident with at least $s+1$*

axes of symmetry different from L, and qIL is a point different from p which is incident with at least one axis of symmetry which is not L. Then every point of L is a translation point.

Proof. Suppose $\mathcal{N}_L$ is the net which corresponds to the regular line L by the dual of Theorem 4.1.1, and suppose that $\mathcal{N}'_L$ is the subnet of $\mathcal{N}_L$ (of the same degree) which is generated (in the sense of Chapter 4) by the lines which correspond to the axes of symmetry meeting L (and different from it) in the quadrangle. Then by Theorem 4.2.1 there only are two possibilities:

(a) $\mathcal{N}'_L$ is an affine plane of order s;

(b) $\mathcal{N}'_L = \mathcal{N}_L$.

Since we supposed that there is a point on L which is incident with at least $s+1$ axes of symmetry different from L, the second possibility holds. It is now clear that every line of $L^\perp \setminus \{L\}$ is an axis of symmetry, and by Theorem 6.7.9, L is also an axis of symmetry. Hence, $\mathcal{S}$ is a translation generalized quadrangle for every point on L. ■

8.2.2 Regularity revisited

The dual of Part (2) of the following lemma is a generalization of a part of Theorem 1.4.2(iv).

Lemma 8.2.2. *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s \neq 1 \neq t$, and let L be a line of $\mathcal{S}$.*

(1) *Assume that each line of $B \setminus L^\perp$ is regular. Then every line of $\mathcal{S}$ is regular.*

(2) *If L is such that every line of $L^\perp \setminus \{L\}$ is regular, then L is also regular.*

Proof. Suppose we are in Case (1). Suppose N is an arbitrary line of $B \setminus L^\perp$. Then since N is a regular line, $\{L, N\}$ is a regular pair (of lines). Hence L is a regular line. Now suppose $L' \neq L$ is a line of $L^\perp$. Let M be a line of $L^\perp$, $L' \not\sim M$. Then since L is regular, $\{L', M\}$ is regular. Now suppose that M' is arbitrary in $B \setminus L^\perp$. Then M' is regular, and hence $\{L', M'\}$ is regular. Hence L' is regular, and then also every line of $\mathcal{S}$. This proves Part (1). Part (2) is easy. ■

8.2.3 Classifications for generalized quadrangles

Theorem 8.2.3. *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s, t), $s \neq 1 \neq t$, and suppose that one of the following conditions holds.*

(1) *There is a regular line L and there are points p and q so that pIL is a point which is incident with at least $s+1$ axes of symmetry different from L, and qIL is a point different from p which is incident with at least one axis of symmetry which is not L.*

(2) *There is a regular line L and there are points p and q so that*

   (a) *pIL is a point which is incident with at least $s+1$ regular lines L_0, $L_1, \ldots, L_s$ different from L, for which there are lines $M_0, M_1, \ldots, M_s$ and points $p_0, p_1, \ldots, p_s$ such that $p_i I L_i \sim M_i \not{I} p_i$ for all i, and such that there is a group of whorls about p_i which fixes M_i and which acts transitively on the points of M_i which are not on L_i, and*

   (b) *qIL is a point different from p which is incident with at least one regular line U which is not L, for which there is a line M and a point u such that $uIU \sim M \not{I} u$, and such that there is a group of whorls about u which fixes M and which acts transitively on the points of M which are not incident with U.*

(3) *$\mathcal{S}$ has two distinct centers of transitivity p and q, and a regular line which is contained in $(pq)^\perp \setminus \{pq\}$ if p and q are collinear.*

(4) *s is even and $\mathcal{S}$ contains a regular line and an elation point p which is not fixed by the group of automorphisms of $\mathcal{S}$.*

(5) *s is odd and $\mathcal{S}$ contains two distinct regular lines and an elation point p which is not fixed by the group of automorphisms of $\mathcal{S}$.*

Then we have the following possibilities:

(i) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4, s)$;*

(ii) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5, s)$;*

(iii) *$t = s^2$, $s = q^n$, where $\mathbf{GF}(q)$ is the kernel of $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$ with q odd, $q \geq 4n^2 - 8n + 2$ and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock;*

(iv) *$t = s^2$, $s = q^n$, where $\mathbf{GF}(q)$ is the kernel of $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$ with q odd, $q < 4n^2 - 8n + 2$ and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

Proof. If (1) is satisfied, then by Lemma 8.2.1 there follows that L is a line of translation points, and the statement follows by Theorem 8.1.1. Case (2) immediately follows from (1) and Theorem 2.3.10.
Next suppose that we are in Case (3). If p and q are non-collinear, then $\mathcal{S}$ is classical by Theorem 7.15.1. Suppose $p \sim q$; then clearly every point of pq is a center of transitivity. Since there is a regular line M in $(pq)^\perp \setminus \{pq\}$, every line of $(pq)^\perp \setminus \{pq\}$ is regular, and by Lemma 8.2.2, pq is also regular. Theorem 2.3.10 implies that every line of $(pq)^\perp \setminus \{pq\}$ is an axis of symmetry, and by Theorem 6.7.9 we can conclude that pq is also an axis of symmetry. The statement follows from Theorem 8.1.1 since pq is a line of translation points.

Now suppose we are in Case (4) of the theorem. If p is mapped onto a point not collinear with p by some automorphism of $\mathcal{S}$, then the statement follows from (3). Hence suppose that p is mapped onto a point $x \sim p \neq x$ by some automorphism of $\mathcal{S}$. Then px is a line each point of which is an elation point, and if the regular line is incident with one of these elation points, then by Theorem 2.3.16 every point incident with pq is a translation point. Now suppose that there is a regular line in $\mathcal{S} \setminus (px)^{\perp}$. Then by transitivity every line of $\mathcal{S} \setminus (px)^{\perp}$ is a regular line, and hence Lemma 8.2.2 implies that every line of $\mathcal{S}$ is regular. Now Theorem 2.3.16 applies. The proof of Case (5) of the theorem is similar by using Theorem 2.3.15. ■

8.3 Reflection

Let us end this chapter with a rather amusing observation in view of Lemma 8.2.2 and the proof of Theorem 8.2.3.

Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, with a line of elation points $[\infty]$, and assume that there is some regular line $L \not\sim [\infty]$.
As each point of $[\infty]$ is an elation point, each line of $\mathcal{S} \setminus [\infty]^{\perp}$ is regular. Whence Lemma 8.2.2 asserts that each line of $\mathcal{S}$ is regular. For $s = t$, it follows that $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4, s)$.
Assume that $t > s$. By Chapter 2, each point incident with $[\infty]$ is a translation point. Hence by Theorem 8.1.1 we have that

(i) $\mathcal{S} \cong \mathcal{Q}(5, s)$ if s is even, or

(ii) $\mathcal{S}^{(\infty)}$ is the translation dual of the point-line dual of a flock GQ, where $(\infty)I[\infty]$ is arbitrary, and each line of $\mathcal{S}$ is regular.

Now apply J. A. Thas [131] to obtain that, in the last case, $\mathcal{S} \cong \mathcal{Q}(5, s)$.

Chapter 9

The Classification Theorem

The main goal of this chapter is to describe the (theoretically) possible subconfigurations of axes of symmetry of a finite abstract GQ.

9.1 The Classification Theorem

Our first lemma is well known.

Lemma 9.1.1. *Suppose $\mathcal{S}$ is a GQ of order s, $s > 1$, which contains a regular pair of distinct non-concurrent lines $\{U, V\}$. Then every line of $\mathcal{S} \setminus \{U, V\}^{\perp\perp}$ intersects at least one line of $\{U, V\}^{\perp\perp}$.*

Proof. The number of lines which are contained in $\{U, V\}^{\perp} \cup \{U, V\}^{\perp\perp}$ or which meet (a line of) the latter line set, is $2(s + 1) + (s + 1)^2(s - 1) = (s + 1)(s^2 + 1)$, and this is the number of lines of the GQ. ■

Lemma 9.1.2. *Suppose $\mathcal{S}$ is an SPGQ of order (s, t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$, and suppose that every point of $\mathcal{S}$ is incident with a constant number, say $k + 1$, $k > 0$, of axes of symmetry. Moreover, suppose that every axis of symmetry hits the base-grid. Then $(t, k) = (s, s)$, i.e., $\mathcal{S} \cong \mathcal{Q}(4, s)$.*

Proof. If $s = t$, then $\mathcal{S} \cong \mathcal{Q}(4, s)$ by Theorem 7.8.1, and $k = s$, see Remark 1.5.6. Suppose by way of contradiction that $s \neq t$, i.e. that $t = s^2$ (by Theorem 7.8.3). We count the number of axes of symmetry of $\mathcal{S} \setminus (\mathcal{L} \cup \mathcal{L}^{\perp})$ in two ways, and distinguish three cases. We obtain the following three equalities, according as $\mathcal{L}^{\perp}$ contains no, respectively one, respectively $s + 1$ axes of symmetry:

(1) $(s + 1)^2 k = (s^3 - s)(k + 1)$;

(2) $(s + 1)(k - 1) + (s + 1)sk = (s^3 - s)(k + 1)$;

(3) $(s + 1)^2(k - 1) = (s^3 - s)(k + 1)$.

None of these equalities is possible if $s > 2$ (by Section 1.6, we do not consider the case $s = 2$). The proof is thus complete. ■

The following (Lenz-Barlotti classification) theorem describes the structure of the possible subconfigurations of axes of symmetry in generalized quadrangles.

Theorem 9.1.3 (Classification Theorem, First Version). *Suppose $\mathcal{S} = (P, B, I)$ is a generalized quadrangle of order (s, t), $s \neq 1 \neq t$. Then we have one of the following possibilities.*

I. *$\mathcal{S}$ contains no axis of symmetry.*

II. *Every axis of symmetry is incident with some fixed point $p \in P$.*

III. *There is a line $L \in B$ which is not an axis of symmetry such that every point $q I L$ is incident with exactly $k+1$ axes of symmetry, k a constant $\in \{0, s-1, s^2-1\}$, and there are no other axes of symmetry.*

IV. *There is an axis of symmetry $L \in B$ such that every point $q I L$ is incident with exactly $k+1$ axes of symmetry, k a constant $\in \{1, s, s^2\}$, and there are no other axes of symmetry.*

V. *Suppose none of the previous cases holds. Define an incidence structure $\mathcal{S}' = (P', B', I')$ as follows.*

- LINES *are of two types:*
  - (1) *the axes of symmetry of $\mathcal{S}$;*
  - (2) *the lines of $\mathcal{S}$ not of Type (1) such that each point of such a line is incident with a line of Type (1).*
- *The* POINTS *of $\mathcal{S}'$ are the points lying on the lines of Type (1), and*
- INCIDENCE *is the restriction of I to $(P' \times B') \cup (B' \times P')$.*

Then $\mathcal{S}'$ is a subGQ of $\mathcal{S}$, every point of $\mathcal{S}'$ is incident with a constant number $k+1$ of axes of symmetry, and one of the following possibilities holds.

- (i) *$k = 0$ and $\mathcal{S}$ has a regular spread $\mathbf{T}_N$ of which any line is an axis of symmetry of $\mathcal{S}$. Furthermore, the group $Aut(\mathcal{S})$ acts transitively on the lines of $\mathbf{T}_N$, $\mathcal{S}' = \mathcal{S}$ and $t = s^2$.*
- (ii) *We have that $k = 1$, and there are two possibilities.*
  - (a) *$\mathcal{S} = \mathcal{S}'$ and $\mathcal{S}$ is of order (s, s^2).*
  - (b) *$\mathcal{S}'$ is a grid with parameters $s+1, s+1$, and hence each line of $\mathcal{S}'$ is an axis of symmetry of $\mathcal{S}$. Moreover, $\mathcal{S}$ is a GQ of order (s, s^2).*
- (iii) *$2 \leq k < t$ and one of the following possibilities holds.*
  - (a) *$\mathcal{S}' = \mathcal{S}$ and $\mathcal{S}$ is of order (s, s^2).*
  - (b) *$k = s$, $\mathcal{S}' \neq \mathcal{S}$, $\mathcal{S}' \cong \mathcal{Q}(4, s)$ and $t = s^2$.*

VI. *Every line of $\mathcal{S}$ is an axis of symmetry and then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$.*

Proof. First suppose that $\mathcal{S}$ is not contained in **I**, and suppose p is a point which is incident with exactly $k+1$ axes of symmetry, $k \geq 0$. Suppose that U is an axis of symmetry not incident with p, such that if $k=0$, then $proj_pU$ is not the axis of symmetry that is incident with p. Then $Aut(\mathcal{S})$ acts transitively on the points of $proj_pU$. Whence *every* point of the line $proj_pU$ is incident with *exactly* $k+1$ axis of symmetry.

STEP 1: The structure $\mathcal{S}'$ is a (possibly thin) generalized quadrangle

The definition of $\mathcal{S}'$ implies that every line of $\mathcal{S}'$ is incident with $s+1$ points of $\mathcal{S}'$. Let R be an arbitrary line of $\mathcal{S}$ which contains at least two different points q and r of $\mathcal{S}'$. Then there is an axis of symmetry through q and also through r, and it follows that there exist axes of symmetry through each point of R. Thus R is completely contained in $\mathcal{S}'$. Since $\mathcal{S}'$ has at least one axis of symmetry, and since all the axes of symmetry of $\mathcal{S}$ do not contain a common point, we can conclude by Theorem 1.3.2 that $\mathcal{S}'$ is a subquadrangle of order (s,t') with $1 \leq t' \leq t$.
Note that also in the cases of **II** and **III** we have that $\mathcal{S}'$ is a subGQ of order (s,t') with $1 \leq t' \leq t$ (this will be used further in the proof).

Suppose $\mathcal{S}$ is not an element of **I**, **II**, **III**, **IV** *and* **VI** *in STEP* 1-2-3-4.
Then we have the following.

It is important to observe that

(*) *Every point of $\mathcal{S}'$ is incident with $k+1$ axes of symmetry of $\mathcal{S}$ with k as above, $0 \leq k \leq t'$.*

We only show this property for the case where $\mathcal{S}'$ is thick, with $s \neq t$. The other cases are left to the reader as an easy exercise (they also follow essentially from the rest of the proof).
We start with noting the following easy property:

(O) *If $x \sim y \neq x$ and if there are axes of symmetry XIx and YIy so that $X \not\sim Y$, then x and y are incident with the same number of axes of symmetry.*

Suppose that p and q are points of $\mathcal{S}'$ which are both incident with the same line U which is not an axis of symmetry. Then there are axes of symmetry V and W of $\mathcal{S}$ so that VIp and WIq. The result then follows from (O). Suppose $p \not\sim q$, and distinguish the following two cases:

- There are axes of symmetry NIp and MIq so that $N \not\sim M$. Suppose that $x \in \mathcal{S}'$ is a point not on a line of $\{N,M\}^{\perp\perp} = \{N_0 = N, N_1, \ldots, N_s = M\}$, which is incident with an axis of symmetry skew to $\{N,M\}^{\perp\perp}$. Define the points $x_i := proj_{N_i}x$ for $i = 0,1,\ldots,s$. Then by (O), each x_i is

incident with the same number of axes of symmetry of $\mathcal{S}$. Since each line of $\{N,M\}^{\perp}$ contains a point of $\{x_0,x_1,\ldots,x_s\}$, the statement now follows from (O).

Now suppose that each axis of symmetry through x intersects $\{N,M\}^{\perp\perp}$. Suppose that p and q are not incident with the same number of axes of symmetry. Then there is at most one axis of symmetry in $\{N,M\}^{\perp}$; otherwise, the group generated by the symmetries about the axes of symmetry in $\{N,M\}^{\perp}$ acts transitively on $\{N,M\}^{\perp}$. Suppose there are axes of symmetry U and U', not contained in $\{N,M\}^{\perp}\cup\{N,M\}^{\perp\perp}$, which are concurrent with different lines of $\{N,M\}^{\perp}$. Let $R\in\{N,M\}^{\perp\perp}$ be concurrent with U. Let rIR be incident with the line R' of $\{N,M\}^{\perp}$ that meets U'. Suppose G is the base-group corresponding to the base-span $\{N,M\}^{\perp\perp}$. Then G acts transitively on R', so r is incident with an axis of symmetry U'', not contained in $\{N,M\}^{\perp}\cup\{N,M\}^{\perp\perp}$ and not intersecting U. The group generated by the symmetries about U and U'' acts transitively on R, and it now follows that there is an automorphism of $\mathcal{S}$ mapping p onto q, contradiction. Hence each axis of symmetry is concurrent with the same line of $\{N,M\}^{\perp}$, implying that $\mathcal{S}$ is a member of **III** or **IV**, contradiction.

– We are not in the previous case. Then NIp and MIq are the only axes of symmetry through, respectively, p and q, and N and M are concurrent. Suppose w is an arbitrary point of $\{p,q\}^{\perp'}\setminus\{N\cap M\}$ (where '$\perp'$' is the restriction of '$\perp$' to $\mathcal{S}'$). Then there is an axis of symmetry WIw. It is now obvious that there is an automorphism of $\mathcal{S}$ mapping p onto q.

Finally, if $pIRIq\neq p$ with R an axis of symmetry, choose a point u in $p^{\perp'}$ or $q^{\perp'}$ such that up or uq is not an axis of symmetry, and apply the preceding observations to obtain that u is incident with the same number of axes of symmetry as p and q. If no such point exists, then every line of $R^{\perp}$ is an axis of symmetry, and (O) applies.

STEP 2: The Case $k=0$

As $k=0$ and as we are not in Case **III**, $\mathcal{S}'$ is a thick GQ which contains non-concurrent axes of symmetry (by Lemma 1.3.5), $\mathcal{S}'$ is an SPGQ of order (s,t'), $t'>1$, and hence $t'\in\{s,s^2\}$ by Theorem 7.8.3. Since $k=0$, there is exactly one axis of symmetry (of $\mathcal{S}$) through each point of $\mathcal{S}'$, so $\mathcal{S}'$ has a spread $\mathbf{T}_N$ each line of which is an axis of symmetry. Observe that $\mathbf{T}_N$ is a Hermitian spread, since every axis of symmetry is regular and since it is clear that $\{L,M\}^{\perp\perp}\subseteq\mathbf{T}_N$ if $L,M\in\mathbf{T}_N$, $L\neq M$. However, a GQ of order s cannot have regular spreads by Lemma 9.1.1, and so $\mathcal{S}=\mathcal{S}'$ and $t=s^2$.

STEP 3: The Case $k=1$

Again, we conclude that $t\in\{s,s^2\}$. Moreover, if $\mathcal{S}$ is of order s, then $\mathcal{S}\cong\mathcal{Q}(4,s)$ by Theorem 7.8.1, contradiction, since in that case every line is an axis of symmetry

(we assumed not to be in **VI**). Hence $t = s^2$. Suppose that $\mathcal{S}' \neq \mathcal{S}$. Then we have two possibilities (by Theorem 1.3.1):

– $\mathcal{S}'$ is a grid with parameters $s+1, s+1$ (in which case any of the $2(s+1)$ lines of $\mathcal{S}'$ is an axis of symmetry of $\mathcal{S}$);

– $\mathcal{S}'$ is a GQ of order s (in which case Lemma 1.3.5 asserts that $\mathcal{S}'$ is an SPGQ of order s, and then Theorem 7.8.1 implies that $\mathcal{S}' \cong \mathcal{Q}(4, s)$).

Suppose we are in the second case, and suppose U and V are arbitrary but non-concurrent axes of symmetry of $\mathcal{S}$. Then $U, V \in \mathcal{S}'$, and every line of $\mathcal{L} = \{U, V\}^{\perp\perp}$ is also an axis of symmetry of $\mathcal{S}$. It is clear that at most one line of $\{U, V\}^{\perp}$ can be an axis of symmetry; otherwise every line of $\{U, V\}^{\perp}$ is, and then by Lemma 9.1.1 no point of $\mathcal{S}' \setminus \{U, V\}^{\perp}$ can be incident with an axis of symmetry, contradiction. Now count the number of axes of symmetry of $\mathcal{S}' \setminus (\mathcal{L} \cup \mathcal{L}^{\perp})$ in two ways to obtain that

$$(s+1)^2 = \frac{(s^3 - s)2}{s},$$

if there is no axis of symmetry in $\{U, V\}^{\perp}$, and then $s = 3$, or that

$$(s+1)s = \frac{(s^3 - s)2}{s},$$

if there is one axis of symmetry in $\{U, V\}^{\perp}$, and then $s = 2$. If $s = 3$, then by Section 1.6 we have that $\mathcal{S} \cong \mathcal{Q}(5, 3)$, a contradiction since every line of $\mathcal{Q}(5, 3)$ is an axis of symmetry. If $s = 2$, then $\mathcal{S}$ is isomorphic to the unique GQ of order $(2, 4)$, namely $\mathcal{Q}(5, 2)$, contradiction.

STEP 4: The Case $k \geq 2$

Since $\mathcal{S}$ is an SPGQ, $t \in \{s, s^2\}$. The case $t = s$ is clearly only possible when $s = t = k$ by Theorem 7.8.1, contradiction.
Suppose that $\mathcal{S}' \neq \mathcal{S}$. By Theorem 1.3.1, $\mathcal{S}'$ is a GQ of order s which is isomorphic to $\mathcal{Q}(4, s)$ by similar arguments as before, and $t = s^2$. Consider a fixed axis of symmetry U of $\mathcal{S}$; since U is regular, we can associate a projective plane Π_U of order s to U by Theorem 4.1.1. Each point of U is incident with k axes of symmetry of $\mathcal{S}'$ which are different from U, and these lines are points of Π_U. Consider the set $\mathcal{B}$ of line spans which contain two distinct axes of symmetry $M, N \sim U$ of $\mathcal{S}$, $M \not\sim N$; then every line of $\{M, N\}^{\perp\perp}$ is an axis of symmetry meeting U, and every span of this form is a line of Π_U.
It is easily seen that the set of all axes of symmetry meeting U and U itself, together with $\mathcal{B}$ and the points of U, with the induced incidence of Π_U, form a subplane Π' of Π_U. Since $k \geq 2$, the plane Π' is not degenerate. However, every line of Π' is incident with exactly $s+1$ points, and so, as Π_U is a projective plane of order s, we have that $\Pi' = \Pi_U$ (see also Chapter 4). We can conclude that every line of $\mathcal{S}'$ meeting U is an axis of symmetry of $\mathcal{S}$.

STEP 5: The Classes III and IV

Every point of the line L is incident with exactly $k+1$ axes of symmetry, $k \in \{0, 1, \ldots, t\}$. Also, since $\mathcal{S}$ is an SPGQ, $t \in \{s, s^2\}$, and we can suppose that $t = s^2$ as before. By the proof of STEP 1, it is clear that the incidence structure $\mathcal{S}'$ as defined above is a (possibly thin) GQ in this case. Suppose $k > 0$ if $\mathcal{S} \in$ **III**, and that $k > 1$ if $\mathcal{S} \in$ **IV**. Then $\mathcal{S}'$ is thick, and $\mathcal{S}'$ is a GQ of order s if $\mathcal{S}' \neq \mathcal{S}$. If we are in the latter case, then it is clear that $k = s - 1$, respectively $k = s$, by Chapter 4 (cf. Theorem 4.2.1). If $\mathcal{S}' = \mathcal{S}$, then $k = s^2 - 1$, respectively $k = s^2$, by that same chapter.

By Theorem 7.14.1, the proof of the theorem now follows. ■

Remark 9.1.4. (i) We will call the classes of Theorem 9.1.3 *symmetry-classes*, since each class reflects (in a certain way) the 'amount of symmetry' of its members. For convenience, we will also call their subclasses (which will be defined in due course) *symmetry-classes*.

(ii) In the determination of the symmetry-classes, we will only consider thick generalized quadrangles 'with symmetry'.

(iii) We will not come back to the Symmetry-Class **VI**, as that class is completely determined by Theorem 9.1.3.

9.2 Symmetry-Class III

The Symmetry-Class III. *There is a line $L \in B$ which is not an axis of symmetry so that every point $q \mathrm{I} L$ is incident with exactly $k+1$ axes of symmetry, $k \in \{0, s-1, s^2-1\}$, and there are no other axes of symmetry. Respectively, we call those classes* **III.1***,* **III.2** *and* **III.3**.

9.2.1 Symmetry-Class III.1

First suppose that $\mathcal{S} \in$ **III.1**. Then we have the following restrictions for $\mathcal{S}$ (which are valid for any member of **III**).

(R1) By Theorem 7.8.3 and Theorem 7.8.1, $\mathcal{S}$ is of order (s, s^2); if $s = t$, then $\mathcal{S} \cong \mathcal{Q}(4, s)$ and all lines are thus axes of symmetry. We also note that s is a prime power (which follows, e.g., from (R2)).

(R2) By Theorem 7.12.1, $\mathcal{S}$ contains $s+1$ subGQ's of order s, all isomorphic to $\mathcal{Q}(4, s)$, and mutually intersecting in the grid Γ with parameters $s+1, s+1$ which is defined by the axes of symmetry of $\mathcal{S}$.

Needless to say, many other restrictions could be stated, such as the fact that any automorphism of $\mathcal{S}$ fixes the grid Γ and stabilizes any of its two reguli of lines.

We do not know examples of **III.1**.

Remark 9.2.1. I personally think that examples of **III.1** need some extra conditions to be determined in a sensible way. Restriction (R2) is already an indication what conditions should be the most natural. We refer to Chapter 12 for more details on that matter.

The Symmetry-Classes **III.2** and **III.3** will be determined in Chapter 11.

9.3 Symmetry-Class IV

The Symmetry-Class IV. *Each element of* **IV** *has an axis of symmetry L so that every point qIL is incident with exactly $k+1$ axes of symmetry, $k \in \{1, s, s^2\}$, and there are no other axes of symmetry.*

Every element of **IV** has order (s, s^2) by Theorem 7.8.3 and Theorem 7.8.1. For $k+1 = 2, s+1, s^2+1$, we will denote the corresponding subclasses of **IV** by **IV.1**, **IV.2** and **IV.3**, respectively.

9.3.1 Symmetry-Class IV.1

Assume that $\mathcal{S} \in \mathbf{IV.1}$. Then we have the same restrictions for $\mathcal{S}$ as in the case of **III.1**. Also, any automorphism of $\mathcal{S}$ fixes the line L and the grid Γ as defined by the axes of symmetry of $\mathcal{S}$ (the reguli of lines of Γ are automatically fixed in this case).

We do not know examples of **IV.1**.

Chapter 10

Symmetry-Class IV.3: Translation Duals of TGQ's which Arise from Flocks

In this chapter, it is shown that each finite translation generalized quadrangle $\mathcal{S}$ which is the translation dual of the point-line dual of a flock generalized quadrangle has a line $[\infty]$ each point of which is a translation point. This leads to the fact that the full group of automorphisms of $\mathcal{S}$ (in the non-classical case) acts 2-transitively on the points of $[\infty]$, and the observation applies concretely to the point-line duals of the (non-classical) Kantor flock generalized quadrangles, to the (non-classical) Roman generalized quadrangles and to the Penttila-Williams generalized quadrangle.
By Chapter 8 it then follows that the generalized quadrangles for which the set of translation points is a line — i.e., those GQ's of Symmetry-Type **IV.3** — are *precisely* the non-classical TGQ's for which the translation dual is the point-line dual of a flock GQ.

We emphasize that, for a long time, it has been thought that every TGQ which is the translation dual of the point-line dual of a flock GQ (or, more generally, every non-classical TGQ) has only one translation point. There are important consequences for the theory of generalized ovoids in $\mathbf{PG}(4n-1,q)$, the study of span-symmetric generalized quadrangles, derivation of flocks of the quadratic cone in $\mathbf{PG}(3,q)$, subtended ovoids in generalized quadrangles, and the understanding of automorphism groups of certain generalized quadrangles. Several problems on these topics will be solved completely.

The present chapter up to Section 10.7 is based on K. Thas [153].

10.1 TGQ's which Arise from Flocks in Even Characteristic

For TGQ's $\mathcal{S}$ of order (q, q^2) that arise from a flock, whereby q is even, we have the following result.

Theorem 10.1.1 (N. L. Johnson [49]). $\mathcal{S} \cong \mathcal{S}^* \cong \mathcal{Q}(5, q)$. ■

Thus we will assume q to be odd in the entire chapter.

10.2 Some Historical Notes

It was a long-standing conjecture that every SPGQ of order s, $s > 1$, is isomorphic to the classical GQ $\mathcal{Q}(4, s)$ (and then s is a prime power). That conjecture was solved affirmatively in Chapter 7. There is no such conjecture for the case $s \neq t$ $(s, t > 1)$ — recall the discussion from the beginning of Chapter 8.

In Chapter 8, it was already mentioned that the dual Kantor semifield flock GQ's have a line each point of which is a translation point, an observation that was first made by S. E. Payne (in another context) in his paper *A garden of generalized quadrangles, Alg., Groups, Geom.* **3** *(1985), 323–354* [79].

We will now sketch the proof of Payne's observation.

Suppose $\mathcal{S} = \mathcal{S}(G, \mathcal{J})$ is a 4-gonal representation of the Kantor flock GQ of order (q^2, q), q odd, with the convention that A_0 is the zero matrix. Then S. E. Payne shows that $\mathcal{S}$ is a TGQ for the line $[A(\infty)]$ (so each point of $[A(\infty)]$ is a center of symmetry). Next, for each $r \neq 0 \in \mathbf{GF}(q)$, the following automorphism of the group G is defined:

$$\theta_r : (\alpha, c, \beta) \longrightarrow (\alpha + \beta K_r^{-1}, c + \frac{1}{4}\beta A_r^{-1}\beta^T, \beta). \tag{10.1}$$

The automorphism θ_r has the properties that

(i) it maps $A(0)$ onto itself,

(ii) $A(\infty)$ onto $A(r)$, while $A(-r)$ is mapped onto $A(\infty)$, and

(iii) $A(t)$ is mapped onto $A(rt/(r+t))$ $(r \neq -t)$.

Also, he proves that the collineation of $\mathcal{S}$ induced by θ_r is a symmetry about $A^*(0)$, from which it follows that $A^*(0)$ is a center of symmetry. There is a major corollary.

Theorem 10.2.1. *Every point of* $(\infty)^\perp$ *is a center of symmetry.* ■

Hence if $[\infty]$ is the line of $\mathcal{S}^D$ which corresponds to (∞) in $\mathcal{S}$, then each point on $[\infty]$ is a translation point, and hence $\mathcal{S}^D$ is an SPGQ with base-span $\{U, V\}^{\perp\perp}$ for each two distinct non-concurrent lines U and V in $[\infty]^\perp$.

10.3 Proof of the Main Theorem

Suppose that $\hat{f}$ is a biadditive and symmetric map of $\mathbb{F}^2 \times \mathbb{F}^2$ onto $\mathbb{F}$, where $\mathbb{F} = \mathbf{GF}(q)$, q odd. Suppose $\hat{g}_t$ is a map of $\mathbb{F}^2$ onto $\mathbb{F}$ so that, for all $d, t, u \in \mathbb{F}$ and $\alpha, \gamma \in \mathbb{F}^2$, the following conditions are satisfied:

(1) $\hat{g}_t(\alpha + \gamma) - \hat{g}_t(\alpha) - \hat{g}_t(\gamma) = \hat{f}(t\alpha, \gamma) = \hat{f}(t\gamma, \alpha)$;

(2) $\hat{g}_{t+u}(\alpha) = \hat{g}_t(\alpha) + \hat{g}_u(\alpha)$;

(3) $\hat{g}_t(\gamma) = 0$, where t is nontrivial, implies that $\gamma = (0, 0)$;

(4) $\hat{g}_t(\gamma(t-d)^{-1}) - \hat{g}_d(\gamma(t-d)^{-1}) + \hat{g}_d(-\gamma(d-u)^{-1}) - \hat{g}_u(-\gamma(d-u)^{-1}) = 0$ implies $\gamma = (0, 0)$ if t, d, u are distinct (this is Condition (10) of [91, Section 10.4]).

Put $G = \mathbb{F}^4 = \{(r, c, b, d) \parallel r, c, b, d \in \mathbb{F}\}$ with coordinatewise addition. Define subgroups in the following way: $B(\infty) = \{(r, 0, 0, 0) \in G \parallel r \in \mathbb{F}\}$; $B^*(\infty) = \{(r, 0, \gamma) \in G \parallel r \in \mathbb{F}, \gamma \in \mathbb{F}^2\}$. For $\gamma \in \mathbb{F}^2$, write $B(\gamma) = \{(-\hat{g}_c(\gamma), c, -c\gamma) \in G \parallel c \in \mathbb{F}\}$. Put $\gamma = (g_1, g_2) \in \mathbb{F}^2$. Define $B^*(\gamma)$ in the usual way (see [82, Section VI]; this will be inessential for our purposes). Then $(\mathcal{J}, \mathcal{J}^*)$ is a 4-gonal family for G, with $\mathcal{J} = \{B(\gamma) \parallel \gamma \in \mathbb{F}^2 \cup \{\infty\}\}$ and $\mathcal{J}^* = \{B^*(\gamma) \parallel \gamma \in \mathbb{F}^2 \cup \{\infty\}\}$. We then have a TGQ $\mathcal{S} = (\mathcal{S}^{(\infty)}, G) = \mathcal{S}(G, \mathcal{J})$ (which satisfies some additional properties, see further) of order (q, q^2). Moreover, any TGQ which is the translation dual of the point-line dual of a flock GQ can be represented in this way, see S. E. Payne [82] for the case of the Roman GQ's, and M. Lavrauw and T. Penttila [61] for the general case (it should be mentioned that their approach is heavily inspired by that of S. E. Payne [82]). Actually, we will show that the TGQ's as defined above are *precisely* the TGQ's for which the translation dual is the point-line dual of a flock GQ.

For convenience, we will work with the point-line dual $\mathcal{S}^D$ of $\mathcal{S}$. This GQ can be represented by the 4-gonal family $\mathcal{J} = \{A(t) \parallel t \in \mathbb{F} \cup \{\infty\}\}$ in the group $H = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$, where $A(t) = \{(\alpha, \hat{g}_t(\alpha), t\alpha) \parallel \alpha \in \mathbb{F}^2\}$, $A(\infty) = \{(\overline{0}, 0, \beta) \parallel \beta \in \mathbb{F}^2\}$, and where the group operation of H is defined by

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \hat{f}(\beta, \alpha'), \beta + \beta').$$

The corresponding groups $A^*(t)$, with $t \in \mathbb{F} \cup \{\infty\}$, are defined by $A^*(t) = \{(\alpha, c, \alpha t) \parallel \alpha \in \mathbb{F}^2, c \in \mathbb{F}\}$ and $A^*(\infty) = \{(\overline{0}, c, \beta) \parallel \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. With this representation, $\mathcal{S}^D$ is a TGQ with base-line $[A(\infty)]$ [82].

Theorem 10.3.1. *Suppose that $\mathcal{S} = \mathcal{S}^{(\infty)}$ is a TGQ with base-point (∞), which is the translation dual of the point-line dual of a flock GQ of order (q^2, q), q odd. Then there is a line $L I (\infty)$ so that every point on L is a translation point. In particular, the group of automorphisms of $\mathcal{S}$ which fix L acts 2-transitively on the points of L.*

Proof. As noted before, any TGQ $\mathcal{S}$ of order (q,q^2), q odd, which is the translation dual of the point-line dual of a flock GQ, can be represented in the aforementioned way. Dualize to obtain $\mathcal{S}^D$, and use the 4-gonal family $\mathcal{J} = \{A(t) \parallel t \in \mathbb{F} \cup \{\infty\}\}$ in the group $H = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$, as above. For arbitrary $v \in \mathbb{F}$, $v \neq 0$, define a collineation θ_v of $\mathcal{S}^D$ as follows:

$$(\alpha, c, \beta) \longrightarrow (\alpha + v^{-1}\beta, c + \hat{g}_{v^{-1}}(\beta), \beta).$$

It is easily checked that θ_v is indeed a nontrivial collineation of $\mathcal{S}^D$ (first note that θ_v induces a nontrivial automorphism of H, and observe that $A(t)$ is mapped onto $A(\frac{t}{1+t/v})$ if $t \neq -v$, that $A(-v)$ is mapped onto $A(\infty)$, and that $A(\infty)$ is mapped onto $A(v)$), and θ_v fixes any point of $(A^*(0))^\perp$. An easy way to see this is the following: each point on $[A(0)]$ is fixed, the point $(\overline{0}, 0, \overline{0})$ is fixed, and since clearly the order of θ_v is p with $q = p^h$ for the prime p, at least one line through $A^*(0)$ different from $[A(0)]$ and $(\overline{0}, 0, \overline{0})A^*(0)$ is fixed. Also, any point of $(\infty)^\perp$ is to be regular; see Remark 10.3.3(iii) for an easy proof. Then by Chapter 2, θ_v indeed is a symmetry about $A^*(0)$. Hence θ_v is a nontrivial symmetry about $A^*(0)$, and $A^*(0)$ is a center of symmetry. Since $\mathcal{S}^D$ is a TGQ with base-line $[A(\infty)]$ (so every point on $[A(\infty)]$ is a center of symmetry), there easily follows that any point of $(\infty)^\perp$ is a center of symmetry, and hence $\mathcal{S}^D$ is a TGQ for every base-line through (∞). There follows that there is some line L through (∞) in $\mathcal{S}^{(\infty)}$ so that each point on L is a translation point, and the theorem easily follows. ■

Note. The fact that each point of $(\infty)^\perp$ is regular in the proof of Theorem 10.3.1 is inessential for that proof. Without this information, it still follows that θ_v, $v \neq 0$, is a collineation of $\mathcal{S}^D$ which is a symmetry about $A^*(0)$.

As a direct corollary, we obtain the following representation method for TGQ's in odd characteristic of which the translation dual arises from a flock.

Theorem 10.3.2. *Suppose that $\hat{f}$ is a biadditive and symmetric map of $\mathbb{F}^2 \times \mathbb{F}^2$ onto $\mathbb{F}$, where $\mathbb{F} = \mathbf{GF}(q)$, q odd. Suppose $\hat{g}_t$ is a map of $\mathbb{F}^2$ onto $\mathbb{F}$ so that, for all $d,t,u \in \mathbb{F}$ and $\alpha, \gamma \in \mathbb{F}^2$, the Conditions (1)–(4) are satisfied. If the GQ $\mathcal{S}$ arises from the 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ as above, then the TGQ $\mathcal{S}$ of order (q,q^2) is the translation dual of the point-line dual of a flock GQ of order (q^2,q), and, conversely, any (non-classical) TGQ of which the translation dual arises from a flock arises in this way .*

Proof. By Theorem 10.3.1, we know that $\mathcal{S}$ has a line of translation points. The theorem now follows from Chapter 8. ■

Remark 10.3.3. (i) If a GQ has non-collinear translation points, then it is well known that it is classical, see e.g. Theorem 7.15.1.

(ii) In Theorem 10.3.1, we could also have stated that $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ is a TGQ which is good at some element $\pi \in \mathcal{O}$ (that corresponds to the line $LI(\infty)$). Since q is odd, by Theorem 2.7.4, $\mathcal{S}^{(\infty)}$ then is the translation dual of the

point-line dual of a semifield flock GQ of order (q^2, q). The converse is 'trivially' true. Below, we will therefore make no distinction between TGQ's in odd characteristic which are good at some line containing the translation point (∞), and TGQ's which are the translation dual of a TGQ of order (q, q^2) arising from a flock, q odd.

(iii) There is also a purely geometrical proof without coordinates, as was noted to us by J. A. Thas, which uses the relation between nets and GQ's with a regular line. We give a sketch of that proof. Suppose $\mathcal{S}^{(\infty)}$ is a TGQ of order (q, q^2), q odd, which is good at its line $L\mathrm{I}(\infty)$. Then there are $q^3 + q^2$ subGQ's of order q, all isomorphic to $\mathcal{Q}(4, q)$, which contain the flag $((\infty), L)$. It follows immediately that L is regular, since $\mathcal{S}^{(\infty)}$ is a TGQ. Now suppose $M \sim L \neq M\mathrm{I}m\mathrm{I}L$. It is clear that if $N \sim L \neq N$ and $N \not\sim M$, then $\{M, N\}$ is a regular pair of lines. Now suppose $U \not\sim M$ is not a line of $L^\perp$. Consider an arbitrary point u of L different from m, and let V be the unique line of $\mathcal{S}$ for which $u\mathrm{I}V \sim U$. Since there are $q^3 + q^2$ subGQ's of $\mathcal{S}$ of order q which contain L, it follows that there is a unique such classical subGQ of $\mathcal{S}$ which contains L, M, V and U (this is also immediate by representing $\mathcal{S}^{(\infty)}$ as $T(\mathcal{O})$, with $T(\mathcal{O})$ good at L). Hence the pair $\{M, V\}$ is regular, and M is a regular line. It follows that every point on L is coregular. Consider any such coregular point $p\mathrm{I}L$. From the regular line L there arises a net $\mathcal{N}_L$, and $\mathcal{N}_L$ is a $\mathcal{P}$-net with $\mathcal{P}$ the parallel class of $\mathcal{N}_L$ defined by p, since the dual of $\mathcal{N}_L$ satisfies the Axiom of Veblen (and so $\mathcal{N}_L$ is the dual of an H^3_s). Hence by Theorem 4.4.3, every point on L is a translation point, and so every line of $L^\perp$ is an axis of symmetry.

(iv) In some sense, Theorem 10.3.1 explains the intrinsic difference between a TGQ which arises from a flock and its translation dual, if the flock is not a Kantor flock or linear flock.

10.4 TGQ's for which the Translation Dual Arises from a Flock, and Span-Symmetric Generalized Quadrangles

Theorem 10.3.1 implies that if $\mathcal{S}$ and L are as in Theorem 10.3.1, then every line of $L^\perp$ is an axis of symmetry, and then for every two non-concurrent lines U and V in $L^\perp$, the GQ $\mathcal{S}$ is span-symmetric with base-span $\{U, V\}^{\perp\perp}$. As a direct corollary of Theorem 10.3.1 and Theorem 8.1.1, we obtain the following important result, which states that the generalized quadrangles with two distinct translation points are *exactly* the TGQ's of which the translation dual is the point-line dual of a flock GQ.

Theorem 10.4.1. *A generalized quadrangle $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t \neq s$, has two distinct collinear translation points if and only if $\mathcal{S}$ is a TGQ which is the translation dual of the point-line dual of a flock GQ. In particular, if s is even, then $\mathcal{S} \cong \mathcal{Q}(5,s)$. If $\mathcal{S}$ has non-collinear translation points, then $\mathcal{S}$ is always of classical type.* ■

Corollary 10.4.2. *The automorphism groups of the dual (non-classical) Kantor GQ's, the (non-classical) Roman GQ's and the Penttila-Williams GQ act* 2*-transitively on the points incident with the line of translation points.*

Proof. This follows immediately by Theorem 10.3.1. ■

Note. For the dual Kantor GQ's, this observation was already made by S. E. Payne in [83], see Chapter 8.

10.5 Determination of Symmetry-Class IV.3

For **IV.3** we have the following.

Theorem 10.5.1. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which has a line L for which each element of $L^\perp$ is an axis of symmetry, and which contains no other axes of symmetry. Then $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (s^2,s) with s an odd prime power, and we have the following possibilities.*

- (i) *If $t = s^2$, $s = q^n$ and q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, and if $q \geq 4n^2 - 8n + 2$, then $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, $\mathcal{F}$ a nonlinear Kantor flock.*
- (ii) *If $t = s^2$, $s = q^n$ and q odd, where $\mathbf{GF}(q)$ is as in (i), and if $q < 4n^2 - 8n + 2$, then $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some nonlinear semifield flock.*

Conversely, if $\mathcal{S} = T(\mathcal{O})$ is a non-classical TGQ of order (s,s^2), s odd, which is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ (or, equivalently, if $\mathcal{O}$ is good at some element), then $\mathcal{S}$ is an element of **IV.3***.*

Proof. Immediately by Theorem 8.1.1 and Theorem 10.3.1. ■

So the elements of **IV.3** are precisely the non-classical TGQ's $T(\mathcal{O})$ which are the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$.

The only known examples of **IV.3** *besides the dual (non-classical) Kantor flock GQ's are the (non-classical) Roman GQ's and the Penttila-Williams GQ. Both examples satisfy the properties of Theorem 10.5.1(ii) (recall that the kernel of both is isomorphic to* $\mathbf{GF}(3)$*).*

10.6 Isomorphisms of Subtended Ovoids in the TGQ's $(\mathcal{S}(\mathcal{F})^D)^*$, $\mathcal{F}$ a Semifield Flock

Suppose $\mathcal{O}$ is an ovoid of a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$. Let x be a point of $\mathcal{O}$, and suppose L is incident with x. Then $|(\mathcal{O} \setminus \{x\}) \cap y^\perp| = t$ for any point $y \neq x$ on L. Denote $(\mathcal{O} \setminus \{x\}) \cap y^\perp$ by $V(\mathcal{O}, x, y)$. We call $\mathcal{O}$ a *translation ovoid w.r.t. the flag* (x, L) if there is an automorphism group $G_{x,L}$ of $\mathcal{S}$ which fixes $\mathcal{O}$ (as a set), which fixes x linewise and L pointwise, and which acts transitively on the points of $V(\mathcal{O}, x, y)$ for each $y \neq x$ on L. We call $\mathcal{O}$ a *translation ovoid w.r.t. the point* x if $\mathcal{O}$ is a translation ovoid w.r.t. the flag (x, M) for each line MIx. Translation ovoids (of generalized $2n$-gons, $n = 2, 3$) were introduced by I. Bloemen, J. A. Thas and H. Van Maldeghem in [12].

Suppose $T(\mathcal{O})$ is a TGQ of order (q^n, q^{2n}) with translation point (∞), and assume that $\mathcal{S}'$ is a subGQ of $T(\mathcal{O})$ of order q^n which contains (∞). Suppose z is a point of $T(\mathcal{O}) \setminus \mathcal{S}'$, and consider $\mathcal{O}_z$. Then it is easy to see that $\mathcal{O}_z$ is a translation ovoid of $\mathcal{S}'$ w.r.t. (∞). Suppose that $\mathcal{O}$ is good at some element π, and suppose that $\pi \in \mathcal{S}'$ (as a line), and let q be odd. Then $\mathcal{S}' \cong \mathcal{Q}(4, q^n)$. It was shown in J. A. Thas [124] and G. Lunardon [65] that there is a canonical connection between translation ovoids of $\mathcal{Q}(4, s)$, s odd, and semifield flocks of the quadratic cone of $\mathbf{PG}(3, s)$. Hence the following result.

Theorem 10.6.1 (J. A. Thas [124]; G. Lunardon [65]). *Translation ovoids of* $\mathcal{Q}(4, s)$*, s odd, w.r.t. a point and semifield flocks of the quadratic cone of* $\mathbf{PG}(3, s)$ *are equivalent objects.* ■

We now have the following result of M. Lavrauw.

Theorem 10.6.2 (M. Lavrauw [59, 60]). *Let $\mathcal{O}$ be an egg of* $\mathbf{PG}(4n-1, q)$ *which is good at the element π, q odd, and consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$. Then all the ovoids of a fixed (arbitrary) subGQ $\mathcal{Q}(4, q^n)$ through the flag $((\infty), \pi)$ which are subtended by a point of Type (2) are isomorphic translation ovoids. Moreover, these ovoids are isomorphic to the ovoid of $\mathcal{Q}(4, q^n)$ arising from the semifield flock which corresponds to the egg $\mathcal{O}$.* ■

Consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ as in Theorem 10.6.2. Then there is a line $[\infty]I(\infty)$ which is a line of translation points by Theorem 10.3.1. Fix the subGQ $\mathcal{S}' = \mathcal{Q}(4, q^n)$ as before. Suppose that x is a point of Type (3) not in $\mathcal{S}'$ in the TGQ $T(\mathcal{O})$ (that is, $x \not\sim (\infty)$). Consider two non-concurrent lines U, V in $[\infty]^\perp \cap \mathcal{S}'$. Then U and V are axes of symmetry of $\mathcal{S}$, and the group G generated by all the symmetries about U and V fixes $\mathcal{S}'$ and $[\infty]$, and acts transitively on the points of $[\infty]$. Since there is a collineation in G which maps (∞) onto the unique point on $[\infty]$ which is collinear with x, there readily follows that x subtends an ovoid which is isomorphic to the translation ovoid subtended by the points of Type (2). We obtain the following result, which completely solves the isomorphism problem for

subtended ovoids in classical subGQ's of order q^n which contain the flag $((\infty), [\infty])$ (that is, $((\infty), \pi)$) in TGQ's of order (q^n, q^{2n}) with a good element, q odd.

Theorem 10.6.3. *Let $\mathcal{O}$ be an egg of $\mathbf{PG}(4n-1, q)$ which is good at the element π, q odd, and consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$. Then all subtended ovoids of a fixed (arbitrary) subGQ $\mathcal{Q}(4, q^n)$ through the flag $((\infty), \pi)$ are isomorphic translation ovoids. Moreover, these ovoids are isomorphic to the ovoid of $\mathcal{Q}(4, q^n)$ arising from the semifield flock which corresponds to the egg $\mathcal{O}$.* ■

As an easy corollary, Theorem 5.4 of M. R. Brown [16] follows (this is the 'isomorphism part' of Theorem 10.6.3 for the (dual) Kantor GQ's).

Note. Using Theorem 10.3.1, it is also easy to prove the 'isomorphism part' of Theorem 10.6.2.

As the subGQ $\mathcal{Q}(4, q^n)$ through $[\infty]$ in Theorem 10.6.3 is arbitrary, we also have the following result.

Theorem 10.6.4. *Two TGQ's $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ and $\mathcal{S}'^{(\infty)'} = T(\mathcal{O}')$ of order (q^n, q^{2n}), q odd, with $\mathcal{O}$ and $\mathcal{O}'$ good at π and π', are isomorphic if and only if the subtended ovoids of a fixed subGQ $\mathcal{Q}(4, q^n)$ of $\mathcal{S}$ through $[\infty]$, where $[\infty]$ corresponds to π, and the subtended ovoids of a fixed subGQ $\mathcal{Q}(4, q^n)$ of $\mathcal{S}'$ through $[\infty]'$, where $[\infty]'$ corresponds to π', are isomorphic translation ovoids of $\mathcal{Q}(4, q^n)$.* ■

10.7 Translation Generalized Quadrangles with Isomorphic Translation Duals

Theorem 10.7.1. *Suppose that $\mathcal{S}^{(x)}$ is a non-classical TGQ which is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q). So q is odd. Then the full automorphism group of $\mathcal{S}^{(x)}$ does not fix x if and only if $\mathcal{S}^{(x)}$ is the point-line dual of a non-classical Kantor flock GQ.*

Proof. Suppose that the translation point x of $\mathcal{S}$ is not fixed by $Aut(\mathcal{S})$. Then as $\mathcal{S}$ is non-classical, all the translation points of $\mathcal{S}$ are incident with the same line $[\infty] I x$. There are $q^3 + q^2$ classical subGQ's of $\mathcal{S}$ of order q which contain $[\infty]$. That line $[\infty]$ is thus fixed by the full automorphism group of $\mathcal{S}$ (and it is the only line with that property), and hence it follows that $[\infty]$ corresponds to the special point (∞) of $\mathcal{S}(\mathcal{F})$, as (∞) is fixed by each automorphism of $\mathcal{S}(\mathcal{F})$, see [92]. But by Theorem 3.4.2, it then follows that $\mathcal{F}$ is a Kantor flock, as the dual net $\mathcal{N}^*_{(\infty)}$ satisfies the Axiom of Veblen. ■

The following corollary characterizes the Kantor semifield flock GQ's.

Theorem 10.7.2. *Suppose that $\hat{f}$ is a biadditive and symmetric map of $\mathbb{F}^2 \times \mathbb{F}^2$ to $\mathbb{F}$, where $\mathbb{F} = \mathbf{GF}(q)$, q odd. Suppose $\hat{g}_t$ is a map of $\mathbb{F}^2$ to $\mathbb{F}$ so that, for all $d, t, u \in \mathbb{F}$ and $\alpha, \gamma \in \mathbb{F}^2$, the following conditions are satisfied:*

(1) $\hat{g}_t(\alpha+\gamma) - \hat{g}_t(\alpha) - \hat{g}_t(\gamma) = \hat{f}(t\alpha, \gamma) = \hat{f}(t\gamma, \alpha)$*;*

(2) $\hat{g}_{t+u}(\alpha) = \hat{g}_t(\alpha) + \hat{g}_u(\alpha)$*;*

(3) $\hat{g}_t(\gamma) = 0$*, where* t *is nontrivial, implies that* $\gamma = (0,0)$*;*

(4) $\hat{g}_t(\gamma(t-d)^{-1}) - \hat{g}_d(\gamma(t-d)^{-1}) + \hat{g}_d(-\gamma(d-u)^{-1}) - \hat{g}_u(-\gamma(d-u)^{-1}) = 0$ *implies* $\gamma = (0,0)$ *if* t, d, u *are distinct.*

Assume also the following additional condition:

(5) *If for* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}^2$ *we have that*

$$0 = \hat{f}(\alpha_1(t-u), \beta_1) = \hat{f}(\alpha_1(t-u), \beta_2) = \hat{f}(\alpha_2(t-u), \beta_1),$$

then $\hat{f}(\alpha_2(t-u), \beta_2) = 0$ *(this is Condition V.2 of [82]).*

Let $\mathcal{S}$ *be the GQ which arises from the* 4*-gonal family* $(\mathcal{J}, \mathcal{J}^*)$ *as before. Then* $\mathcal{S}^D \cong \mathcal{S}(\mathcal{F})$ *with* $\mathcal{F}$ *a Kantor flock, and conversely.*

Proof. By Section 10.3 and Theorem 10.3.1, $\mathcal{S}$ is a TGQ for which $(\mathcal{S}^*)^D$ is a flock GQ, say $\mathcal{S}(\mathcal{F})$. Condition (5) is exactly the condition for $\mathcal{S}^D$ to be a flock GQ, see [82] (in fact, Condition (5) implies that $\mathcal{S}^D$ satisfies Property (G) at its point (∞), and then Theorem 2.7.4 implies that it is a flock GQ). The theorem now follows from Theorems 10.3.1 and 10.7.1. ■

Another corollary of Theorem 10.7.1 is the following.

Theorem 10.7.3. *Suppose* $T(\mathcal{O})$ *is a TGQ of order* (q, q^2)*,* q *odd, where* $\mathcal{O}$ *is good at some element. Then* $T(\mathcal{O})$ *is the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *if and only if* $\mathcal{F}$ *is a Kantor flock.*

Proof. This follows immediately by Theorem 10.3.1, Theorem 10.7.1 and the fact that as $\mathcal{O}$ is good at some element, $T(\mathcal{O})^*$ is the point-line dual of a flock GQ. ■

Note. The latter result was first obtained by L. Bader et al. [4].

There are some immediate corollaries, which we state as a theorem.

Theorem 10.7.4. (i) *Suppose* $\mathcal{S} = T(\mathcal{O})$ *is a TGQ of order* (q, q^2)*,* q *odd, where* $\mathcal{O}$ *is good at some element. Then* $T(\mathcal{O}) \cong T(\mathcal{O}^*)$ *if and only if* $\mathcal{S}$ *is the point-line dual of a Kantor flock GQ.*

(ii) *Suppose* $\mathcal{S} = T(\mathcal{O})$ *is a TGQ of order* (q, q^2)*,* q *odd, which is the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$*. Then* $\mathcal{S}$ *is isomorphic to its translation dual if and only if* $\mathcal{F}$ *is a Kantor flock.*

Proof. Immediate. ■

There is another interesting corollary.

Corollary 10.7.5. *Suppose $\mathcal{S}$ is the Roman GQ of order (q, q^2), $q = 3^h$, $h > 3$ (so $\mathcal{S}$ is the translation dual of the point-line dual of the flock GQ $\mathcal{S}(\mathcal{F})$, with $\mathcal{F}$ a nonlinear Ganley flock). If $Aut(\mathcal{S})$ is the full automorphism group of $\mathcal{S}$, then*

$$|Aut(\mathcal{S})| = q^6(q+1)(q-1)2h.$$

For $q = 27$, $|Aut(\mathcal{S})| = q^6(q+1)(q-1)8h$.

Proof. This follows immediately by Theorem 10.7.1 and the fact that the full automorphism group of $\mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a Ganley flock has size $q^6(q-1)2h$ when $q > 27$, and size $q^6(q-1)8h$ when $q = 27$, cf. Chapter 3. ∎

10.8 Automorphism Groups Revisited

In this section, it is our objective to obtain a lower bound for the size of the full automorphism group of a non-classical TGQ which is the translation dual of a TGQ that arises from a flock. The bound will be sharp, since equality will hold for the Roman GQ's.

Suppose that $\mathcal{S}$ is a non-classical TGQ of order (q, q^2), q odd, which is the translation dual of a TGQ of order (q, q^2) arising from a flock. By Theorem 10.3.1, $\mathcal{S}$ has a line $[\infty]$ of translation points, and there are $q^3 + q^2$ classical subGQ's of order q all containing $[\infty]$. Consider a fixed subGQ $\mathcal{S}' \cong \mathcal{Q}(4, q)$ of order q through the line $[\infty]$, and note that each symmetry of $\mathcal{S}$ about a line of $\mathcal{S}' \cap [\infty]^\perp$ fixes $\mathcal{S}'$. It is then clear that $Aut(\mathcal{S})_{\mathcal{S}'} =: H$ (that is, the stabilizer of $\mathcal{S}'$ in the automorphism group of $\mathcal{S}$) acts transitively on the ordered pairs (x, y) of points in $\mathcal{S}' \setminus [\infty]$ for which $x \sim y$, xy not intersecting $[\infty]$. Hence we obtain that

$$q^4(q+1) \quad \text{divides} \quad |H|.$$

Note that H fixes $[\infty]$. Fix an ordinary quadrangle $\mathcal{A}$ in $\mathcal{S}'$ which contains $[\infty]$ as a side, and suppose $[\infty], U, V, W$ are the lines of $\mathcal{A}$, so that $U \not\sim [\infty]$. Consider the action of the elementwise stabilizer $H(\mathcal{A})$ of $\mathcal{A}$ in H on the lines of $X := \{U, [\infty]\}^{\perp\perp} \setminus \{U, [\infty]\}$. By Theorem 10.3.1, V and W are axes of symmetry of $\mathcal{S}$ (and $\mathcal{S}'$), and the group G generated by the symmetries about V and W fixes $\mathcal{S}'$ and every line of $\{V, W\}^\perp$, and is isomorphic to $\mathbf{SL}(2, q)$. Hence the kernel of the action of $H(\mathcal{A})$ on the lines of X has a subgroup of order $q - 1$ (recall that the action of G on $\mathcal{S}' \setminus \{V, W\}^{\perp\perp}$ is faithful). Hence

$$q^4(q+1)(q-1) \quad \text{divides} \quad |H|.$$

Let $v = V \cap [\infty]$ and $w = W \cap U$. Then the group $\mathbf{W}(v, w)$ of whorls about v and w has size $|\mathbb{K}| - 1$, where $\mathbb{K}$ is the kernel of the TGQ. This group clearly fixes $\mathcal{S}'$, and acts semiregularly on X, see Theorem 1.3.4. Thus

$$(|\mathbb{K}| - 1)q^4(q+1)(q-1) \quad \text{divides} \quad |H|.$$

It is also clear that each subGQ of order q of $\mathcal{S}$ which contains $[\infty]$ has an $Aut(\mathcal{S})$-orbit of size at least q^2, since $Aut(\mathcal{S})$ acts transitively on the pairs of non-concurrent lines in $[\infty]^\perp$, and hence

$$|Aut(\mathcal{S})| \geq q^6(q+1)(q-1)(|\mathbb{K}|-1).$$

Recall at this point that, if x_1, x_2, x_3, x_4 are four collinear points of $\mathbf{PG}(n,q)$, with $|\{x_1, x_2, x_3, x_4\}| \geq 3$, then by $(x_1, x_2; x_3, x_4)$ we denote the usual *cross-ratio* given by

$$\frac{r_3 - r_1}{r_3 - r_2} : \frac{r_4 - r_1}{r_4 - r_2},$$

where the r_i's, $i = 1, 2, 3, 4$, are non-homogeneous coordinates of the x_i's on the line through x_1, x_2, x_3, x_4.
We also recall that, if a semilinear automorphism θ of $\mathbf{PG}(n,q)$ (i.e. $\theta \in \mathbf{P\Gamma L}(n+1,q)$) preserves the cross-ratio of all 4-tuples of points on at least one line, then θ is a linear automorphism of $\mathbf{PG}(n,q)$ (i.e. $\theta \in \mathbf{PGL}(n+1,q)$). Note that if a semilinear automorphism θ of $\mathbf{PG}(n,q)$ fixes some $\mathbf{PG}(k,q)$ in $\mathbf{PG}(n,q)$, $k > 0$, elementwise, then θ preserves the cross-ratio, and hence $\theta \in \mathbf{PGL}(n+1,q)$.

By Theorem 2.4.4, we can consider $Aut(\mathcal{S})_{(\infty)}$, for any fixed point $(\infty)I[\infty]$, as a group of automorphisms of $\mathbf{PG}(4n, q')$, where $q = {q'}^n$, which fixes the corresponding egg $\mathcal{O} \subset \mathbf{PG}(4n-1, q')$. Moreover, since the groups from the previous arguments, if restricted to $\mathbf{P\Gamma L}(4n+1, q')_{\mathcal{O}}$, all fix at least one line of $\mathbf{PG}(4n-1, q')$ pointwise, it is clear that

$$|Aut(\mathcal{S})_{(\infty)} \cap \mathbf{PGL}(4n+1, q')| \geq q^6(q-1)(|\mathbb{K}|-1).$$

We now show that the latter bound is sharp. Suppose that $(\mathcal{S}(\mathcal{F})^D)^*$, with $\mathcal{F}$ a Ganley flock, is a Roman GQ of order (q, q^2), $q > 27$. Then $|Aut((\mathcal{S}(\mathcal{F})^D)^*)_{(\infty)} \cap \mathbf{PGL}(4n+1, q')| = q^6(q-1)2$, by the previous section and the preceding arguments, Chapter 3 and Theorem 2.4.4.

10.9 Derivation of Semifield Flocks, BLT-Sets and Automorphisms

The following theorem solves the isomorphism problem of derivation for the flocks of a BLT-set in the semifield case.

Theorem 10.9.1. *Suppose that $\mathcal{S}^{(\infty)}$ is a TGQ which is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q). Suppose $\{\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_1, \ldots, \mathcal{F}_q\}$ is the BLT-set of $q+1$ flocks derived from $\mathcal{F}$. Then all these flocks are isomorphic if and only if $\mathcal{F}$ is a Kantor flock. If $\mathcal{F}$ is not a Kantor flock, then $\mathcal{F}_1, \ldots, \mathcal{F}_q$ are all isomorphic, but non-isomorphic to $\mathcal{F}$.*

Proof. If all these flocks are isomorphic, then this implies that $Aut(\mathcal{S}(\mathcal{F})^D)$ acts transitively on the points of the line $[\infty]$ which corresponds to the point (∞) of $\mathcal{S}(\mathcal{F})$. Hence $[\infty]$ is a line of translation points, and so $\mathcal{F}$ is a Kantor flock by Theorem 10.7.1. The theorem easily follows. ∎

10.10 Configurations of Translation Points

In this section, we state results concerning configurations of translation points in the known TGQ's. They are direct corollaries (or restatements) of the results obtained in this chapter.

Theorem 10.10.1. *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q, q^2), $q > 1$ and q odd, which is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. Then $T(\mathcal{O})$ has two distinct translation points if and only if $\mathcal{F}$ is a Kantor flock.* ∎

Theorem 10.10.2. (i) *Assume that $\mathcal{S}$ is a known non-classical TGQ which is the translation dual of a TGQ which arises from a flock, i.e. suppose that $\mathcal{S}$ is, respectively, a non-classical Roman GQ, the Penttila-Williams GQ or a non-classical dual Kantor GQ. Then $\mathcal{S}$ contains a line L of translation points.*

(ii) *Let $\mathcal{F}_G$, respectively $\mathcal{F}_{PW}$, be a nonlinear Ganley flock, respectively Penttila-Williams flock. Then $\mathcal{S}(\mathcal{F}_G)^D$ (which is the translation dual of the Roman GQ), respectively $\mathcal{S}(\mathcal{F}_{PW})^D$ (which is the translation dual of the Penttila-Williams GQ), has exactly one translation point.*

(iii) *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t \neq s$, which contains two distinct translation points. If s is even, then $\mathcal{S} \cong \mathcal{S}^* \cong \mathcal{Q}(5, s)$.* ∎

10.11 Symmetry-Class II.3

The Symmetry-Class II.3. *The elements of* **II.3** *are precisely the TGQ's which have one translation point.*

Here, we only summarize the known GQ's with one translation point (as a direct result of previous observations).

(i) For $\mathcal{O}$ a non-classical oval, respectively non-classical ovoid, in $\mathbf{PG}(2, q)$, respectively $\mathbf{PG}(3, q)$, the GQ $T_2(\mathcal{O})$, respectively $T_3(\mathcal{O})$, has precisely one translation point.

(ii) A $T_2(\mathcal{O})^D$ of order q has one translation point if and only if $\mathcal{O}$ is a translation oval in $\mathbf{PG}(2, q)$ which is not a conic (and note that q is therefore even). In that case, $T_2(\mathcal{O})^D \cong T_2(\mathcal{O})$.

(iii) Suppose $\mathcal{F}$ is a Ganley flock or a Penttila-Williams flock. Then $\mathcal{S}(\mathcal{F})^D$, which is a GQ of order (q, q^2), has exactly one translation point (where $q > 9$ in the case $\mathcal{F}$ is a Ganley flock).
More generally, if $\mathcal{F}$ is derived from a nonlinear semifield flock which is not a Kantor flock, the same property holds for $\mathcal{S}(\mathcal{F})^D$.

(iv) Each non-classical TGQ of order (s, t), $s \neq 1 \neq t$ and s even, has one and only one translation point by Theorem 8.1.1.

Chapter 11

A Characterization Theorem and a Classification Theorem

The two main theorems of this chapter will appear to be very valuable for the rest of the present work.

11.1 GQ's of Order (s, s^2) with s^3+s^2 SubGQ's of Order s Through Some Fixed Line

The proof of the next theorem was essentially obtained in the previous chapter.

Theorem 11.1.1. *Suppose L is a line of the GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$ and $s \neq t$, such that there are $s^3 + s^2$ distinct subGQ's of $\mathcal{S}$ of order s, all isomorphic to $\mathcal{Q}(4,s)$, and all containing L.*

(1) *If s is even, then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

(2) *If s is odd, then $\mathcal{S}$ is the translation dual — w.r.t. an arbitrary translation point — of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (s^2,s), s an odd prime power, and $\mathcal{S}$ contains a line of translation points.*

Proof. Suppose that L is so that there are s^3+s^2 such subGQ's of $\mathcal{S}$ of order s, all containing L. Then it follows immediately that L is regular, since any pair of lines in $\mathcal{S}$ of the form $\{L,U\}$, $L \not\sim U$, is contained in a $\mathcal{Q}(4,s)$ subGQ (and any line of $\mathcal{Q}(4,s)$ is regular). Now suppose $M \sim L \neq MImIL$. It is clear that if $N \sim L \neq N$ and $N \not\sim M$, then $\{M,N\}$ is a regular pair of lines. Let $U \not\sim M$ be a line not contained in $L^{\perp}$. Consider an arbitrary point u of L different from m, and let V be the unique line of $\mathcal{S}$ for which $uIV \sim U$. Since there are s^3+s^2 subGQ's which are isomorphic to $\mathcal{Q}(4,s)$ and which contain L, there is a (necessarily) unique such subGQ of $\mathcal{S}$ which contains L, M, V and U (cf. Theorems 1.3.1 and 1.3.2). Hence

the pair $\{M, V\}$ is regular, and M is a regular line. It follows that every point on L is coregular. Consider any such coregular point pIL. From the regular line L there arises a net $\mathcal{N}_L$, and $\mathcal{N}_L$ is a $\mathcal{P}$-net (cf. Chapter 4) with $\mathcal{P}$ the parallel class of $\mathcal{N}_L$ defined by p, as the dual of $\mathcal{N}_L$ clearly satisfies the Axiom of Veblen (and so $\mathcal{N}_L$ is isomorphic to the dual of H_s^3 by Theorem 4.4.1). Hence by Theorem 4.4.3, every point on L is a translation point, and every line of $L^\perp$ is an axis of symmetry. The result now follows from Theorem 8.1.1. ∎

CONJECTURE. *Suppose L is a line of the GQ $\mathcal{S}$ of order (s, s^2), $s \neq 1$, such that there are $s^3 + s^2$ distinct subGQ's of $\mathcal{S}$ of order s, all containing L. Then we have the following possibilities for $\mathcal{S}$:*

(1) *$\mathcal{S} \cong T_3(\mathcal{O})$ for some ovoid $\mathcal{O}$ of $\mathbf{PG}(3, s)$ (in particular $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, s)$);*

(2) *s is even and $\mathcal{S}$ is the point-line dual of a flock GQ of order (s, s^2);*

(3) *s is odd and $\mathcal{S}$ contains a line of translation points (so $\mathcal{S}$ is the translation dual — w.r.t. any translation point — of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (s^2, s)).*

11.2 A Classification Theorem

The following theorem is very strong.

Theorem 11.2.1. *Let $\mathcal{S}$ be a GQ of order (s, t), $s \neq 1 \neq t$, and suppose $L \sim M \neq L$ are axes of symmetry. Moreover, suppose that N is an axis of symmetry of $\mathcal{S}$ which does not meet L or M. Then we have one of the following:*

(i) *$t = s$ and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4, s)$;*

(ii) *$t = s^2$, s is even, and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, s)$;*

(iii) *$t = s^2$ and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (s^2, s), s an odd prime power, and $\mathcal{S}$ contains a line of translation points.*

Proof. Let L, M and N be as above, and define U as the line through $M \cap L$ which intersects N. We can suppose that $s \neq t$, since otherwise $\mathcal{S} \cong \mathcal{Q}(4, s)$ by Theorem 7.8.1. By Theorem 9.1.3, we also know that each point of U is incident with $k + 1$ axes of symmetry, where

$$k \in \{s - 1, s, s^2 - 1, s^2\}.$$

If $k = s^2$, then every point on U is a translation point, and Theorem 8.1.1 applies. Now put $k = s^2 - 1$. Then by Theorem 6.7.9, every point on U is a translation point, and Theorem 8.1.1 applies. Next suppose that $k \in \{s - 1, s\}$. Then by STEP 5 of

the proof of Theorem 9.1.3, the set of points on the axes of symmetry of $\mathcal{S}$ form a classical subGQ $\mathcal{S}'$ of $\mathcal{S}$ of order s (isomorphic to $\mathcal{Q}(4, s)$). Fix an arbitrary point u on U. In $\mathcal{S}'$, the symmetries about any three distinct lines through u different from U generate a group of elations about u of size s^3 which extends to a group of automorphisms of $\mathcal{S}$ of size s^3 (cf. Chapter 6). In $\mathcal{S}'$, U is also an axis of symmetry. Let θ be a symmetry about U in $\mathcal{S}'$, and let L_1, L_2, L_3 be distinct lines through u, all different from U and contained in $\mathcal{S}'$. Then we can write θ as

$$\theta = \theta_1\theta_2\theta_3,$$

where θ_i is a symmetry of $\mathcal{S}'$ about L_i. Now consider an extended automorphism θ' of $\mathcal{S}$ of θ (i.e. θ' induces θ in $\mathcal{S}'$). Then θ' fixes all lines through u in $\mathcal{S}$, and all lines of $U^\perp \cap \mathcal{S}'$. Hence, by Chapter 4 (cf. Theorem 4.2.6), θ' is a symmetry about U. It now readily follows that U is also an axis of symmetry of $\mathcal{S}$, and we can suppose that $k = s$. Also, through any two non-concurrent axes of symmetry V, W in $U^\perp$ there are $s + 1$ classical subGQ's of order s (one of which is $\mathcal{S}'$), mutually intersecting in the points and lines of $\{V, W\}^{\perp\perp} \cup \{V, W\}^\perp \subseteq \mathcal{S}'$, see Theorem 7.12.1. Counting all the classical subGQ's of order s which arise in this way, we easily obtain $s^3 + 1$ such GQ's (including $\mathcal{S}'$).
Suppose L_1, L_2, L_3, u are as above, and suppose G is the group of size s^3 of automorphisms of $\mathcal{S}$ which is generated by the symmetries about L_1, L_2, L_3. Suppose G_* is an arbitrary G-orbit of the permutation group $(P \setminus u^\perp, G)$, where P is the point set of $\mathcal{S}$. Now define the incidence structure $\mathcal{S}(G_*) = \mathcal{S}^* = (P^*, B^*, I^*)$ as follows.

- The POINTS of P^* are of three types:
  - (1) the point u;
  - (2) the points of G_*;
  - (3) any point which is incident with a line of $\mathcal{S}'$ through u.
- We have two types of LINES:
  - (a) the s + 1 lines through u in $\mathcal{S}'$;
  - (b) the lines of $\mathcal{S}$ which intersect a line of the first type and contain at least one point of G_*.
- The INCIDENCE RELATION $I^* \subseteq I$ is the restriction of I to $(P^* \times B^*) \cup (B^* \times P^*)$.

Then one easily observes that $\mathcal{S}^*$ has $(s+1)(s^2+1)$ points and the same number of lines, and that any line has $s+1$ points. Hence $\mathcal{S}^*$ is a subGQ of $\mathcal{S}$ of order s. Furthermore, it clearly is a TGQ with translation point u. By considering all the G-orbits in $P \setminus u^\perp$, s such GQ's arise (including $\mathcal{S}'$) for a fixed u. In total, $s^2 - 1$ distinct subGQ's of order s arise, all distinct from $\mathcal{S}'$, which intersect $\mathcal{S}'$ in the

lines of $\mathcal{S}'$ through some point xIU, together with the points incident with those lines (we will call such a subGQ 'a subGQ of Type (2)'). Hence U is contained in $s^3 + s^2$ distinct subGQ's of $\mathcal{S}$ of order s.
Consider an arbitrary subGQ $\mathcal{S}''$ of $\mathcal{S}$ of Type (2). Then $\mathcal{S}''$ is a TGQ with respect to the point zIU for which $\mathcal{S}'' \cap \mathcal{S}'$ is the induced subgeometry of $\mathcal{S}$ on the lines of $\mathcal{S}'$ through u.
We distinguish two cases.

(1) s IS ODD. Consider two non-concurrent lines U' and U'' in $U^\perp$; then there are $s+1$ distinct subGQ's of order s containing U, U' and U'', and thus $\{U', U''\}$ is a regular pair of lines by Theorem 1.3.2. It follows easily that U is a regular line. As U is contained in s^3+s^2 distinct subGQ's of $\mathcal{S}$ of order s, the dual net $\mathcal{N}_U^*$ satisfies the Axiom of Veblen, and hence $\mathcal{N}_U^* \cong H_s^3$ by Theorem 4.4.1. Hence, if $\mathcal{S}''$ is an arbitrary subGQ of $\mathcal{S}$ of order s containing the line U, then $(\Pi_U)_{\mathcal{S}''}$ (the projective plane of order s which arises from the regular line U in $\mathcal{S}''$) is Desarguesian. As s is odd, the result follows from Corollary 4.4.4 and Theorem 11.1.1.

(2) s IS EVEN. Let $\mathcal{S}''$ be a subGQ of $\mathcal{S}$ of Type (2), and suppose that z is the point on U so that $\mathcal{S}'' \cap \mathcal{S}'$ is the induced subgeometry of $\mathcal{S}$ defined by $z^\perp$ in $\mathcal{S}'$. Then each line of $\mathcal{S}''$ through z is regular (in $\mathcal{S}''$), as $\mathcal{S}''$ is a TGQ with translation point z. Fix such a line ZIz which is different from U. Let Γ be an arbitrary grid with parameters $s+1, s+1$ which is contained in $\mathcal{S}''$ and which contains Z and U. Then Γ is contained in $s+1$ subGQ's of $\mathcal{S}$ of order s (which contain U), s of which are not of Type (2) — hence which are isomorphic to $\mathcal{Q}(4,s) \cong W(s)$ (recall that s is even). Denote these subGQ's by $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_s$, and suppose $\{p_1, p_2, p_3\}$ is an arbitrary triad of points which are contained in Γ. Then as $\mathcal{S}_i \cong W(s)$ for all $i = 1, 2, \ldots, s$, $\{p_1, p_2, p_3\}$ (clearly) is unicentric in $\mathcal{S}_i$. As $\mathcal{S}$ has order (s, s^2), we also know that $|\{p_1, p_2, p_3\}^\perp| = s+1$, and so we infer that $\{p_1, p_2, p_3\}$ is also (uni)centric in $\mathcal{S}''$. By Theorem 1.5.3, we conclude that $\mathcal{S}'' \cong W(s)$, and thus U is contained in $s^3 + s^2$ subGQ's of $\mathcal{S}$ which are isomorphic to $\mathcal{Q}(4,s) \cong W(s)$, and Theorem 11.1.1 applies.

The proof is complete. ■

For the even characteristic case, there is an important corollary.

Corollary 11.2.2. *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and suppose $L \sim M \neq L$ are axes of symmetry. Moreover, suppose that N is an axis of symmetry of $\mathcal{S}$ which does not meet L or M, and let s be even. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$.* ■

We end this section with the following conjecture:

CONJECTURE. *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and suppose $L \sim M \neq L$ are regular lines. Moreover, suppose that N is a regular line of $\mathcal{S}$ which does not meet L or M. Then we have one of the following:*

(i) *$t = s$ and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$;*

(ii) $t = s^2$, *s is even, and* $\mathcal{S}$ *is isomorphic to* $\mathcal{Q}(5, s)$*;*

(iii) $t = s^2$ *and* $\mathcal{S}$ *is the translation dual of the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *of order* (s^2, s)*,* s *an odd prime power, and* $\mathcal{S}$ *contains a line of translation points.*

11.3 Determination of Symmetry-Classes III.2 and III.3

We can now infer that

Theorem 11.3.1. *The Symmetry-Classes* **III.2** *and* **III.3** *are empty.*

Proof. Since an element $\mathcal{S}$ of **III.2** or **III.3** contains axes of symmetry L, M and N so that $L \sim M \neq L$ and such that N does not meet L or M, we can apply Theorem 11.2.1 to obtain that $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (s^2, s), s an odd prime power, and that $\mathcal{S}$ contains a line of translation points, contradiction. ■

11.4 Symmetry-Class IV.2

Regarding the Symmetry-Class **IV.2**, we can immediately proceed as in Theorem 11.3.1, and thus we have the following.

Theorem 11.4.1. *The Symmetry-Class* **IV.2** *is empty.*

Proof. See the proof of Theorem 11.3.1. ■

Chapter 12

The Symmetry-Class V

We start this chapter with defining the Symmetry-Class $\mathbf{V}$.

The Symmetry-Class V. *We define the subclasses of* $\mathbf{V}$ *as follows:*

(i) **V.1** *is* $\mathbf{V}.1$ *of Theorem 9.1.3;*

(ii) **V.2** *is* $\mathbf{V}.2.(a)$*;*

(iii) **V.3** *is* $\mathbf{V}.2.(b)$*;*

(iv) **V.4** *corresponds to* $\mathbf{V}.3.(a)$ *and, finally,*

(v) **V.5** *corresponds to* $\mathbf{V}.3.(b)$.

Whence if $\mathcal{S}'$ is the (possibly thin) subGQ of order (s, t') which is 'generated' by the axes of symmetry (as described in Theorem 9.1.3), and if k is the constant so that every point of $\mathcal{S}'$ is incident with $k+1$ axes of symmetry (of $\mathcal{S}$), then we have

V.1. $k = 0$ and $\mathcal{S}$ has a Hermitian spread $\mathbf{T}_N$ of which any line is an axis of symmetry;

V.2. $k = 1$ and $\mathcal{S}' = \mathcal{S}$, where $\mathcal{S}$ is of order (s, s^2), $s > 1$;

V.3. $k = 1$ and $\mathcal{S}'$ is a grid with parameters $s+1, s+1$; $s > 1$. Moreover, $\mathcal{S}$ is a GQ of order (s, s^2);

V.4. $2 \leq k < t$ and $\mathcal{S}' = \mathcal{S}$, where $\mathcal{S}$ is of order (s, s^2), $s > 1$;

V.5. $k = s$ and $\mathcal{S}' \neq \mathcal{S}$ is isomorphic to $\mathcal{Q}(4, s)$, $s > 1$. Moreover, $\mathcal{S}$ is a GQ of order (s, s^2).

12.1 The Symmetry-Classes V.1, V.2 and V.4

The complete determination of **V.2** and **V.4** follows from Chapter 11. However, in this section we will give a totally different proof, based essentially on K. Thas [154], that is valuable for any element of $\mathbf{V.1} \cup \mathbf{V.2} \cup \mathbf{V.4}$, thus providing more insight in the structure of those generalized quadrangles *for which the axes of symmetry cover the quadrangle*. We will show that **V.1**, **V.2** and **V.4** are empty, that is, every element of each of these classes is automatically classical. The essential ingredient is that each element of $\mathbf{V.1} \cup \mathbf{V.2} \cup \mathbf{V.4}$ has a 'special' spread, forcing the GQ to have many classical subGQ's.

Lemma 12.1.1. *Any thick element of* $\mathcal{V} := \mathbf{V.1} \cup \mathbf{V.2} \cup \mathbf{V.4}$ *contains a spread each line of which is an axis of symmetry.*

Proof. Suppose $\mathcal{S}$ is an arbitrary thick element of $\mathcal{V}$ of order (s,t). Then $t = s^2$ by Theorem 7.8.1, and every point of $\mathcal{S}$ is incident with at least one axis of symmetry. Fix a base-span $\mathcal{L}$ for the SPGQ $\mathcal{S}$. Then there is a classical subGQ $\mathcal{S}'$ of order s containing the base-grid by Theorem 7.12.1. Now fix an axis of symmetry M of $\mathcal{S}$ not in $\mathcal{S}'$. Then $\mathbf{T}(\mathcal{L}, M)$ (cf. Section 7.13) is the desired spread. ■

We hence have

Theorem 12.1.2 (See also K. Thas [154]). $\mathcal{V} = \mathbf{V.1} \cup \mathbf{V.2} \cup \mathbf{V.4}$ *is empty.*

Proof. Suppose $\mathcal{S}$ is an element of $\mathcal{V}$ of order (s, s^2), $s > 1$, and fix a spread $\mathbf{T}$ each line of which is an axis of symmetry. By Section 7.10 every two non-concurrent lines U and V in $\mathbf{T}$ are contained in $s+1$ classical subGQ's of order s which mutually intersect in the geometry of $\{U,V\}^{\perp\perp} \cup \{U,V\}^{\perp}$. If we now count the number ν of classical subGQ's of order s which arise from all the base-spans of the SPGQ $\mathcal{S}$ which are contained in $\mathbf{T}$, then by Lemma 9.1.1 we obtain

$$\nu = \frac{s^3(s^3+1)(s+1)}{(s+1)s} = (s^3+1)s^2.$$

If we count the number of pairs $(L, \mathcal{S}')$, where L is a line of $\mathcal{S}$ and $\mathcal{S}'$ a subGQ as above which contains L, in two ways, then we obtain that

$$\overline{N}(s^2+1)(s^3+1) = (s^3+1)s^2(s+1)(s^2+1),$$

where $\overline{N}$ is the average number of such subGQ's through a line of $\mathcal{S}$. But then $\overline{N} = s^3 + s^2$, which implies that *each* line of $\mathcal{S}$ is contained in *precisely* $s^3 + s^2$ such subGQ's (by Lemma 4.2.5 and straightforward counting). By Lemma 4.2.5, it now easily follows that every centric triad of lines of $\mathcal{S}$ is contained in a proper subGQ of order s, and by Theorem 1.5.5(i), the theorem follows[1]. ■

Corollary 12.1.3. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which admits a spread each line of which is an axis of symmetry. Then $\mathcal{S} \cong \mathcal{Q}(5,s)$.* ■

[1]Note that Theorem 1.5.5(i) does not require the subGQ's to be classical.

12.2 Solution of a Conjecture of W. M. Kantor

The following is an immediate corollary of Theorem 12.1.2, and is in fact the solution of the natural analogue of the 'SPGQ-conjecture' for GQ's of order (s, s^2), $s > 1$. It is also the solution of a recent conjecture of W. M. Kantor[2] (*Some Conjectures Concerning Generalized Quadrangles*, W. M. Kantor, January 2001, Private communication [55]). The proof of it was published in K. Thas [154], where also a theory for *strongly irreducible collineation groups* of GQ's is introduced, in which the aforementioned result is then applied, cf. the next section.

Theorem 12.2.1 (K. Thas [154]). *A generalized quadrangle of order (s,t), $s \neq 1 \neq t$, is isomorphic to $\mathcal{Q}(5,s)$ if and only if there are three distinct axes of symmetry U, V and W for which $U \cap \{V,W\}^{\perp\perp} = \emptyset$ (where we view U and $\{V,W\}^{\perp\perp}$ as point sets).* ■

Remark 12.2.2. In *Some Conjectures Concerning Generalized Quadrangles*, W. M. Kantor also describes an 'SPGQ-conjecture' for generalized quadrangles of order (s^2, s^3), $s > 1$.

12.3 Irreducible Groups of Generalized Quadrangles

In this section, we introduce the notion of '(strongly) irreducible (automorphism) group' of a GQ. We call an automorphism group G of a generalized quadrangle $\mathcal{S}$ *irreducible* if it satisfies the following conditions:

(a) G does not fix a point of $\mathcal{S}$;

(b) G does not fix a line of $\mathcal{S}$;

(c) G does not fix an ordinary quadrangle.

Note that if G acts irreducibly on $\mathcal{S}$, and p is an arbitrary point of $\mathcal{S}$, then G *cannot fix the set of lines incident with p*. We call this property and its dual (*). An irreducible automorphism group G is *strongly irreducible*, if the following additional condition is satisfied:

(c') G does not fix a subGQ of order (s', t'), $s', t' \geq 1$, $(s', t') \neq (s, t)$.

One notes that Condition (c) is a special case of Condition (c').

Remark 12.3.1. (i) It should be remarked that the notion of (strongly) irreducible group is a self-dual one.

(ii) Note also that the notion of '(strongly) irreducible group' of a GQ is the natural analogue of the 'same' notion for projective planes, see e.g. C. Hering and M. Walker [39].

[2] Which I posed independently.

(iii) Other relevant work (with a slightly different approach to irreducible automorphism groups of GQ's) can be found in M. Walker [167].

We are now able to classify all thick generalized quadrangles admitting a strongly irreducible automorphism group, and having an axis of symmetry:

Theorem 12.3.2. *Let $\mathcal{S}$ be a GQ of order (s,t), $s,t > 1$, admitting a strongly irreducible automorphism group. If $\mathcal{S}$ has an axis of symmetry, then $\mathcal{S}$ is isomorphic to one of $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$.*

Proof. Suppose G is a strongly irreducible automorphism group acting on $\mathcal{S}$, and let L be an axis of symmetry. By Property (*), L^G contains some line M which does not intersect L. So $\mathcal{S}$ is an SPGQ, and $t \in \{s, s^2\}$. If $t = s$, the theorem follows from Theorem 7.8.1. Let $t = s^2$. Then clearly, L^G contains axes of symmetry L, M, N so that the conditions of Theorem 12.2.1 are satisfied (this also follows from Theorem 9.1.3). Now apply that theorem. ∎

Remark 12.3.3. (i) Theorem 12.3.2 is in fact a stronger version of the analogue for generalized quadrangles of the main result of C. Hering and M. Walker [39] for projective planes.

(ii) For $t < s^2$, Theorem 12.3.2 also holds for irreducible automorphism groups.

12.4 Symmetry-Class V.3: Grid-Symmetric Generalized Quadrangles

For an element $\mathcal{S}$ of **V.3***, the axes of symmetry of $\mathcal{S}$ form a grid Γ with parameters $s+1, s+1$, and $\mathcal{S}$ is a GQ of order (s,s^2), $s > 1$.*

(R1) By the same arguments as usual, $\mathcal{S}$ is of order (s,s^2), s a prime power.

(R2) By Theorem 7.12.1, $\mathcal{S}$ contains $s+1$ subGQ's of order s, all isomorphic to $\mathcal{Q}(4,s)$, which mutually intersect in the grid Γ. For each regulus in Γ, there is an automorphism group of $\mathcal{S}$ which

- is isomorphic to $\mathbf{SL}(2,s)$;
- which acts semiregularly on the points of $\mathcal{S} \setminus \Gamma$;
- which fixes the regulus elementwise.

The following result is due to J. A. Thas and S. E. Payne [133] for the even case, M. R. Brown [17] for the odd case, and L. Brouns, J. A. Thas and H. Van Maldeghem [15] for a uniform proof of both cases.

Theorem 12.4.1 ([133],[17],[15]). *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which contains a proper subGQ $\mathcal{S}'$ isomorphic to $\mathcal{Q}(4,s)$, so that every subtended ovoid of $\mathcal{Q}(4,s)$ is an elliptic quadric. Then $\mathcal{S} \cong \mathcal{Q}(5,s)$.* ∎

Very recently, W. M. Kantor [57] has showed that each GQ which satisfies the conditions of **V.3** — he calls such a GQ "grid-symmetric" — necessarily is isomorphic to $\mathcal{Q}(5,s)$. The Properties (R1) and (R2) are crucial in his elegant proof; he considers an arbitrary but fixed subGQ $\mathcal{S}' \cong \mathcal{Q}(4,s)$ which contains Γ, takes a point x of the GQ outside $\mathcal{S}'$, and shows that, using the group theoretical observation of (R2), the subtended ovoid $\mathcal{O}_x$ of $\mathcal{S}'$ is an elliptic quadric:

Theorem 12.4.2 (W. M. Kantor [57]). *Any grid-symmetric generalized quadrangle of order (s,s^2), $s>1$, is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. Suppose $\mathcal{S}$ is a grid-symmetric generalized quadrangle of order (s,s^2) with base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$; so each line of Γ is an axis of symmetry. Let G, respectively G', be the group generated by the symmetries about the lines of $\mathcal{L}$, respectively $\mathcal{L}'$. Then $G \cong G' \cong \mathbf{SL}(2,s)$ by Theorem 7.12.1. For each point x not in Ω, $\Omega \cup x^G$ is the point set of a subGQ $\mathcal{S}(x) \cong \mathcal{Q}(4,s)$ of $\mathcal{S}$ of order s by Theorem 7.12.1. If $U \in \mathcal{L}^\perp$, and H_U is the group of symmetries about U, then H_U acts on $\mathcal{S}(x)$ (it fixes each line of $\mathcal{S}(x)$ meeting U), and so GG' also acts on $\mathcal{S}(x)$. For each $V \in \mathcal{L}$ and $W \in \mathcal{L}^\perp$, $H_U \cap H_W = \{\mathbf{1}\}$, and as H_V and H_W normalize each other, they commute. So G and G' also commute and are both isomorphic to $\mathbf{SL}(2,s)$, so that

$$\langle G, G' \rangle = GG' \cong \mathbf{\Omega}^+(4,s)$$

acts in the natural way on the space $\Pi = \mathbf{PG}(4,s)$ underlying $\mathcal{S}(x)$. As x lies on a line of Π which joins the point m of $\Pi \setminus \mathcal{S}(x)$ perpendicular to $\langle \Omega \rangle$ and a point r of $\langle \Omega \rangle \setminus \Omega$, the stabilizer $(GG')_x$ fixes r. As $(GG')_r \cong \mathbf{\Omega}(3,s) \cong \mathbf{PSL}(2,s)$ has no proper subgroup of index $gcd(2,s-1)$, and as $(GG')_r$ permutes the $gcd(2,s-1)$ points of $\mathcal{S}(x)$ on the line $\langle m, r \rangle$, it follows that

$$(GG')_x = (GG')_r \cong \mathbf{PSL}(2,s).$$

Consider any point $y \notin \Omega \cup x^G$ and the resulting point orbit y^G and subGQ $\mathcal{S}(y)$. Then $K := (GG')_y \cong \mathbf{\Omega}(3,s) \cong \mathbf{PSL}(2,s)$ acts on Ω, and on the ovoid $\mathcal{O}_y$ of $\mathcal{S}(x)$ subtended by y. Via the Klein correspondence, $\mathcal{O}_y$ produces a symplectic spread $\mathbf{T}$ of $\mathbf{PG}(3,s)$, and hence a symplectic translation plane π of order s^2 of which the kernel contains $\mathbf{GF}(s)$. Under this correspondence, the group $GG' \cong \mathbf{\Omega}^+(4,s)$ produces a subgroup of $\mathbf{GL}(4,s)$ (acting on the vector space $\mathbf{AG}(4,s)$) which is isomorphic to $G \times G'$ (see D. E. Taylor [112] for more information), and that has a subgroup R isomorphic to $\mathbf{PSL}(2,s)$ or $\mathbf{SL}(2,s)$ produced by $(GG')_y$ (and R preserves the spread $\mathbf{T}$). If s is odd, then $R \cong \mathbf{SL}(2,s)$, since all involutions of $G \times G'$ lie in its center, and since $\mathbf{PSL}(2,s)$ *has* involutions but no center. All such planes are known by [103, 168]; they are of Hall, Hering, Otto-Schaeffer or Walker type [103, 168]. It follows that only the Desarguesian ones are symplectic (which boils down to looking at the corresponding ovoids of the Klein quadric, and observing that all except the elliptic quadric span $\mathbf{PG}(5,s)$). Hence π is Desarguesian, and $\mathcal{O}_y$ is an elliptic quadric. The theorem now follows from Theorem 12.4.1. ■

Note that the group $\mathbf{\Omega}^+(4,q)$ can be defined as the subgroup of $\mathbf{PGL}(5,q) \cap Aut(\mathcal{Q}(4,q))$ which stabilizes a $\mathcal{Q}^+(3,q) \subset \mathcal{Q}(4,q)$ (that is, a $\mathcal{Q}(3,q)$ in GQ notation), and which is generated by the root groups of $\mathcal{Q}^+(3,q) \subset \mathcal{Q}(4,q)$. So $\mathbf{\Omega}^+(4,q) = \mathbf{SL}(2,q)\mathbf{SL}(2,q)$, both latter groups in their natural action on $\mathcal{Q}(4,q)$ as a base-group corresponding to a fixed regulus of $\mathcal{Q}^+(3,q)$. See D. E. Taylor [112] for more details.

12.5 Symmetry-Class V.5

If $\mathcal{S}$ is an element of **V.5***, then the axes of symmetry of $\mathcal{S}$ form a proper subGQ $\mathcal{S}' \cong \mathcal{Q}(4,s)$ of $\mathcal{S}$ of order s (i.e., each line of $\mathcal{S}'$ is an axis of symmetry of $\mathcal{S}$).*

Each element of **V.5** is thus of order (s, s^2), s a prime power, and by Chapter 11 (cf. Theorem 11.2.1), we can conclude that $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$. However, in this section we want to give a (more) direct approach without the explicit construction of subGQ's of Type (2) as in the proof of Theorem 11.2.1. This approach will (essentially) also work in a more general setting, see Theorem 12.5.5 and its proof.

Let us now obtain, in an elementary (combinatorial) fashion,

Theorem 12.5.1. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which contains a proper subGQ $\mathcal{S}'$ isomorphic to $\mathcal{Q}(4,s)$, such that every symmetry of $\mathcal{S}'$ about a line of $\mathcal{S}'$ can be extended to a symmetry of $\mathcal{S}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. By preceding considerations, we know that if L and M are non-concurrent lines of $\mathcal{S}'$, then there are $s+1$ classical subGQ's ($\cong \mathcal{Q}(4,s)$) of order s which mutually intersect in the points and lines of $\{L,M\}^{\perp\perp} \cup \{L,M\}^\perp$. Fix a point $x \in \mathcal{S} \setminus \mathcal{S}'$, and consider the ovoid $\mathcal{O}_x$ of $\mathcal{S}'$ which is subtended by x. Consider three arbitrary but distinct points u, v, w on $\mathcal{O}_x$. Let U, V, W and U', V', W' be lines in $\mathcal{S}'$ such that $UIuIU'$, $U' \neq U$, $VIvIV'$, $V' \neq V$ and $WIwIW'$, $W' \neq W$, and such that $U \not\sim V$ and $W \in \{U,V\}^\perp$, and $V' \not\sim W'$ and $U' \in \{V', W'\}^\perp$.

We will show first that such lines indeed exist. Suppose $\{u,w\}^\perp \cap \mathcal{S}' = \{u_0, u_1, \ldots, u_s\}$. Then v cannot be collinear with all points $u_0, u_1, \ldots, u_s$; if s is odd, each span of non-collinear points in $\mathcal{Q}(4,s)$ has size 2, while if s is even, we obtain a contradiction by Theorem 1.9.2 and the fact that each point of $\mathcal{Q}(4,s)$ is regular. Suppose $u_i \not\sim v$ for some $i \in \{0,1,\ldots,s\}$. Then put $U = uu_i$, $W = wu_i$ and $V = proj_v W$. Now suppose that v is collinear with each point of $\{u_0, u_1, \ldots, u_s\} \setminus \{u_i\}$. Then it is straightforward to see that $v \sim u_i$, contradiction. Hence there is some point $u_j \neq u_i$, $0 \leq j \leq s$, not collinear with v, and U', V', W' are defined similarly as U, V, W.

Define $\mathcal{S}^1$, respectively $\mathcal{S}^2$, as the unique (classical) GQ of order s through x and U, V, W, respectively x and U', V', W', and note that these GQ's are different. (Note also that, for example, $\mathcal{S}^1$ contains the grid with parameters $s+1, s+1$ which contains U, V and W.) The GQ's $\mathcal{S}^1$ and $\mathcal{S}^2$ intersect in the geometry

defined by the points and lines of $x^\perp \cap \mathcal{S}^1 = x^\perp \cap \mathcal{S}^2$, see Theorem 1.3.2. But

$$\{U,V\}^\perp \cap \{V',W'\}^\perp \subseteq \mathcal{S}^1 \cap \mathcal{S}^2 \cap \mathcal{S}' = \mathcal{O}_x \cap \mathcal{S}^1 = \mathcal{O}_x \cap \mathcal{S}^2$$

(the latter objects viewed as point sets), implying that the conic $\{U,V\}^\perp \cap \{V',W'\}^\perp$ of $\mathcal{S}'$ is completely contained in $\mathcal{O}_x$.
Hence, through any three distinct points of $\mathcal{O}_x$ there goes a conic of $\mathcal{S}'$ which lies on $\mathcal{O}_x$, and hence $\mathcal{O}_x$ is an elliptic quadric (see, e.g., [133]). Thus, every subtended ovoid of $\mathcal{S}'$ is an elliptic quadric, and by Theorem 12.4.1, $\mathcal{S} \cong \mathcal{Q}(5,s)$. ■

Theorem 12.5.2. *The Symmetry-Class* **V.5** *is empty.* ■

Remark 12.5.3. Theorem 12.5.1 is also a direct corollary of the much stronger Theorem 12.4.2. The proof of Theorem 12.5.1 is more elementary though.

The following result is due to L. Brouns, J. A. Thas and H. Van Maldeghem [15].

Theorem 12.5.4 ([15]). *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which contains a proper subGQ $\mathcal{S}'$ isomorphic to $\mathcal{Q}(4,s)$, such that every linear automorphism of $\mathcal{S}'$ can be extended to an automorphism of $\mathcal{S}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.* ■

Note that the 'linear automorphism group' of $\mathcal{Q}(4,q)$ is isomorphic to that of $W(q)$, which is defined by $Aut(W(q)) \cap \mathbf{PGL}(4,q) = \mathbf{PGSp}(4,q)$.
We now generalize Theorem 12.5.1 and Theorem 12.5.4 to Theorem 12.5.5. We did/do not bother to adapt the proof of L. Brouns, J. A. Thas and H. Van Maldeghem of Theorem 12.5.4 (which explicitly uses the order of $\mathbf{PGSp}(4,q)$), and choose for a completely different approach that essentially boils down to constructing subGQ's (via the action of $\mathbf{SL}(2,q)$). We do not have to use the fact that the symmetries about the lines of $\mathcal{Q}(4,q)$ generate $\mathbf{PSp}(4,q) = Aut(W(q)) \cap \mathbf{PSL}(4,q)$ (although we mention a short proof in the even case using this); our approach is rather more locally.

Theorem 12.5.5. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which contains a proper subGQ $\mathcal{S}'$ isomorphic to $\mathcal{Q}(4,s)$, such that every symmetry about a line of $\mathcal{S}'$ can be extended to an automorphism of $\mathcal{S}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. By Section 1.6, we can suppose that $s > 3$ in the rest of the proof.
Assume that $Aut^+(\mathcal{S}')$ is the group of automorphisms of $\mathcal{S}'$ generated by the symmetries about all lines of $\mathcal{S}'$. Then $Aut^+(\mathcal{S}') \cong Aut(W(s)) \cap \mathbf{PSL}(4,s) = \mathbf{PSp}(4,s)$, and $\mathbf{PSp}(4,s)$ is a group of index $gcd(2,s-1)$ in $\mathbf{PGSp}(4,s)$. So if s is even, $\mathbf{PGSp}(4,s) \cong \mathbf{PSp}(4,s)$, so that the even case follows from Theorem 12.5.4 (it will also follow from the rest of the proof).

As $\mathcal{S}$ is a GQ of order (s,s^2), each span of non-collinear points of $\mathcal{S}$ has size 2 (cf. Theorem 1.4.1). Now fix a point $x \in \mathcal{S} \setminus \mathcal{S}'$, and consider the ovoid $\mathcal{O}_x$ of $\mathcal{S}'$ which is subtended by x. Then there is at most one other point y which also subtends $\mathcal{O}_x$. Hence the group extension H^+ of $Aut^+(\mathcal{S}')$ in $Aut(\mathcal{S})$ can have size

at most $2|Aut^+(\mathcal{S}')|$ (by Theorem 1.3.3 and Theorem 1.3.1). We distinguish two cases (according as the extension is proper or not).

(a) $|H^+| = |Aut^+(\mathcal{S}')|$. So H^+ acts faithfully on $\mathcal{S}'$. Fix two non-concurrent lines L and M of $\mathcal{S}'$. As we know, the group G generated by the symmetries about L and M has size $s^3 - s$ (in its action on $\mathcal{S}'$), and is isomorphic to $\mathbf{SL}(2, s)$ (in its action on $\mathcal{S}'$). Suppose that H is the subgroup of H^+ which induces G on $\mathcal{S}'$.
We have that

$$H \cong G \cong \mathbf{SL}(2, s).$$

Note that H, respectively G, acts regularly on the points of $\mathcal{S}' \setminus \Omega$, where Ω is the set of points incident with the lines of $\{L, M\}^{\perp\perp}$.

The semiregular case. We first suppose that H acts semiregularly on (the points of) $\mathcal{S} \setminus \Omega$. If G' is the base-group of $\mathcal{S}'$ corresponding to the base-span $\{L, M\}^{\perp}$, and H' is the subgroup of H^+ which induces G' on $\mathcal{S}'$, then we may suppose that H' acts semiregularly on $\mathcal{S} \setminus \Omega$, as $Aut^+(\mathcal{S}')$ acts transitively on the pairs of non-concurrent lines of $\mathcal{S}'$. The following property is now clear:

(N) *H and H' normalize each other.*

The reason for this is the fact that they normalize each other in the subGQ (and that we are in the semiregular case).

Suppose $\Delta(H) = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_s\}$ is the set of H-orbits (of points) in $\mathcal{S} \setminus \mathcal{S}'$, and that $\Delta(H') = \{\Lambda'_1, \Lambda'_2, \ldots, \Lambda'_s\}$ is the set of H'-orbits in $\mathcal{S} \setminus \mathcal{S}'$. By (N), we have that H acts on $\Delta(H')$, and that H' acts on $\Delta(H)$.
Now define $Y = H'_{\Lambda_i}$, where i is arbitrary in $\{1, 2, \ldots, s\}$. Then $|Y| \geq (s+1)(s-1)$. So if Y' is the group which is induced by Y on $\{L, M\}^{\perp}$, then $|Y'| \geq (s+1)(s-1)/gcd(s-1, 2)$, and Y' is a subgroup of $\mathbf{PSL}(2, s)$.

Now recall Dickson's classification of the subgroups of $\mathbf{PSL}(2, q)$, with $q = p^h$, p a prime (see [47, Hauptsatz 8.27, p. 213]). We list the possible subgroups $K \leq \mathbf{PSL}(2, q)$, as follows:

(i) K is an elementary abelian p-group;

(ii) K is a cyclic group of order k, where k divides $\frac{q\pm1}{r}$, with $r = gcd(q-1, 2)$;

(iii) K is a dihedral group of order $2k$, where k is as in (ii);

(iv) K is the alternating group A_4, where $p > 2$, or $p = 2$ and $h \equiv 0 \mod 2$;

(v) K is the symmetric group S_4, where $p^{2h} - 1 \equiv 0 \mod 16$;

(vi) K is the alternating group A_5, where $p = 5$, or $p^{2h} - 1 \equiv 0 \mod 5$;

(vii) K is a semidirect product of an elementary abelian group of order p^m ($m \neq 0$) with a cyclic group of order k, where k divides $p^m - 1$ and $p^h - 1$;

(viii) K is a $\mathbf{PSL}(2, p^m)$, where m divides h, or a $\mathbf{PGL}(2, p^n)$, where $2n$ divides h.

If $Y' \not\cong \mathbf{PSL}(2, s)$, then it is easy to exclude all the possible cases if $s \notin \{3, 5, 7, 11\}$, and if we are not in Case (vii). The latter case is only possible if $m = h$, and if $k = p^h - 1$. Clearly, this last case is not possible if H' does not act transitively on $\Delta(H)$ (by comparing the orders). If H' acts transitively on $\Delta(H)$, then $|Y'| \geq (s^2 - 1)/2$, also a contradiction. If $s \in \{3, 5, 7, 11\}$, each ovoid of $\mathcal{Q}(4, s)$ is an elliptic quadric by Theorem 1.9.3, so then the result follows by Theorem 12.4.1. We may conclude that $Y' \cong \mathbf{PSL}(2, s)$, so that $Y \cong \mathbf{SL}(2, s)$ and $Y = H'$. Hence H and H' have exactly the same orbits in $\mathcal{S} \setminus \Omega$. Define $\Phi = \langle H, H' \rangle$. As Φ acts faithfully on $\mathcal{S}'$, we have that $\Phi \cong \mathbf{\Omega}^+(4, s)$, so that

$$|\Phi| = \frac{(s+1)^2 s^2 (s-1)^2}{gcd(s-1, 2)}.$$

Note that for each Φ-orbit Λ' in $\mathcal{S} \setminus \mathcal{S}'$, $|\Lambda'| = s^3 - s$.

It is straightforward to see that for each line U of $\mathcal{S}$ which is concurrent with a line of $\{L, M\}^{\perp\perp}$ and not contained in $\mathcal{S}'$, there is a Φ-orbit Λ so that U is incident with at least two distinct points of Λ. For, suppose that this is not the case. Then U meets each of the s Φ-orbits in $\mathcal{S} \setminus \mathcal{S}'$ in precisely 1 point. Let Λ_i be such an arbitrary orbit, and put $u = U \cap \Lambda_i$. Then if RIu is a line which meets the point set of $\{L, M\}^{\perp\perp}$, $R \neq U$, then

$$|\Phi_{U,x,R}| \geq \frac{s-1}{gcd(s-1, 2)}.$$

But each element of $\Phi_{U,x,R}$ fixes each point of U, implying that $\Phi_{U,x,R}$ fixes one and the same subGQ of order s pointwise, a contradiction as soon as $\frac{s-1}{gcd(s-1,2)} > 2$. So $s < 4$ or $s = 5$. But we do not consider these cases by Section 1.6 and Theorem 1.9.3. So for each line U of $\mathcal{S}$ which is concurrent with a line of $\{L, M\}^{\perp\perp}$ and not contained in $\mathcal{S}'$, there is indeed a Φ-orbit Λ containing at least two distinct points of U.

Let U be arbitrary, and suppose Λ is as in the preceding paragraph. Suppose that one of these points is x. Then

$$|\Phi_{x,U,y}| \geq \frac{s-1}{gcd(s-1, 2)},$$

where $y \sim x$ is incident with some line of $\{L, M\}^{\perp\perp}$ but not with U. Suppose z is the point of U which is incident with some line of $\{L, M\}^{\perp\perp}$. We have that either $\Phi_{x,U,y}$ acts semiregularly on the points of $U \setminus \{x, z\}$, or that there is a nontrivial $\theta \in \Phi_{x,U,y}$ which fixes each point of Ω (recall that H, respectively H', acts as $\mathbf{PSL}(2, s)$ on the lines of $\{L, M\}^{\perp\perp}$, respectively $\{L, M\}^{\perp}$). In the last case (cf. Theorems 1.3.1 and 1.3.3), θ fixes a subGQ $\mathcal{S}_\theta$ of $\mathcal{S}$ of order s elementwise.

Suppose that this is not the case. If $\Phi_{x,U,y}$ acts *transitively* on $U \setminus \{x,z\}$, the line U is completely contained in $\Lambda \cup \Omega$ (recall that $|U \cap \Lambda| \geq 2$). Define an incidence structure $\mathcal{S}'' = (P', B', I')$ as follows.

- Lines. The elements of B' are the lines of $\mathcal{S}''$ and they are of two types:
  (1) the lines of $\{L,M\}^{\perp} \cup \{L,M\}^{\perp\perp}$;
  (2) the lines of $\mathcal{S}$ which contain a point of Λ and a point of a line of $\{L,M\}^{\perp\perp}$.
- Points. The elements of P' are the points of $\mathcal{S}''$ and they are the points of $\Omega \cup \Lambda$.
- Incidence. Incidence I' is the induced incidence.

Then it is easy to see that $\mathcal{S}''$ is a subGQ of $\mathcal{S}$ of order s, as there is a line completely contained in $\Lambda \cup \Omega$, and using the transitivity of Φ on Λ (and the fact that $|\Lambda| = s^3 - s$). Hence if $\Phi_{x,U,y}$ always (for each choice of L, M and U) acts regularly or not semiregularly on $U \setminus \{x,z\}$, we can conclude the following property:

(C) *If Z and Z' are arbitrary non-concurrent lines of $\mathcal{S}'$, and z is a point of $\mathcal{S} \setminus \mathcal{S}'$, then there is a subGQ of $\mathcal{S}$ of order s containing $\{Z,Z'\}^{\perp\perp}$ (as a grid), and the point z.*

Now take over the corresponding part of Theorem 12.5.1.

Suppose $\Phi_{x,U,y}$ acts semiregularly, but not regularly, on the points of $U \setminus \{x,z\}$; so $|\Phi_{x,U,y}| = \frac{s-1}{2}$. Then one easily obtains a contradiction by considering the action of $\Phi_{x,U,y}$ on $U \setminus (\Lambda \cup \Omega)$ ($\Phi_{x,U,y}$ has one orbit of size 1 and one of size $\frac{s-1}{2}$ in $\Lambda \cap U$, and acts regularly on the $\frac{s-1}{2}$ points of $U \setminus (\Lambda \cup \Omega)$ in some Φ-orbit Λ_i; now interchange the roles of Λ and Λ_i).

Hence (C) is satisfied, and the corresponding part of Theorem 12.5.1 can be taken over.

The non-semiregular case. Suppose that H does not act semiregularly on the points of $\mathcal{S} \setminus \Omega$ (so that it does not act semiregularly on the points of $\mathcal{S} \setminus \mathcal{S}'$). Since $H \cong \mathbf{SL}(2,s)$, we know (cf. Theorem 7.8.2) that H contains precisely $s+1$ subgroups of order s (which are Sylow p-subgroups, $s = p^h$), and we denote these groups by $H_0, H_1, \ldots, H_s$. Suppose $x \in \mathcal{S} \setminus \mathcal{S}'$ is fixed by some nontrivial element of H, and let $\Lambda = \Lambda(x)$ be the H-orbit of x. Let p be a point of Λ, and consider the $s+1$ H_i-orbits O_i which contain p. Then every point of O_i is also a point of Λ. By the fact that $H_iH_j \cap H_k = \{\mathbf{1}\}$ for distinct i,j,k (and the semiregular action of each H_r on $\mathcal{S} \setminus \mathcal{S}'$), we can take over the counting arguments of the proof of Lemma 7.9.2 in the obvious way, to obtain that

$$|\Lambda| \geq s^3 - s.$$

But as x is fixed by a nontrivial element of H, $|H| \geq 2(s^3 - s)$, contradicting the fact that $H \cong \mathbf{SL}(2, s)$. So this case cannot occur.

(b) $|H^+| = 2|Aut^+(\mathcal{S}')|$. Clearly, one may assume that $|H| = 2(s^3 - s)$, because otherwise the result follows from Part (a). Hence H contains a subgroup N of size 2 that acts trivially on $\mathcal{S}'$, so that $H/N \cong G \cong \mathbf{SL}(2, s)$. Exactly the same argument as in Section 7.10 now produces the contradiction.

The result follows. ■

Chapter 13

Recapitulation of the Classification Theorem

This chapter contains a table which states a short description of each of the symmetry-classes. All the known generalized quadrangles — sometimes slightly more general — are in the appropriate class. Still, we emphasize that w.r.t. the classification in its full generality, the table is not complete; the semifield flock GQ's and their translation duals, for example, are completely classified relative to the symmetry-classes by the earlier considerations (so the table is focused more on 'concrete' examples). Other information will be clear from the table.
We end the chapter by a classification result for general SPGQ's, that follows from the Lenz-Barlotti classification obtained in this work.

13.1 The Classification Theorem, Second Version

The notation of Theorem 9.1.3 is used. In the table, '(D)' stands for 'description'; '(M)' stands for 'known members'. We stress the fact that the symmetry-classes are defined to be *disjoint*. Only thick GQ's are considered, and below, the parameters s and t are as usual. We do not mention the hypothetically possible parameters when the class is empty.

Symmetry-Class I

Symmetry-Class I. (D) *$\mathcal{S}$ contains no axis of symmetry.*

(M) $H(3,q^2)$, $H(4,q^2)^D$, $W(q)$ with q odd, $H(4,q^2)$, $T_3(\mathcal{O})^D$ with $\mathcal{O}$ an ovoid of $\mathbf{PG}(3,q)$, $\mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a flock of the quadratic cone in $\mathbf{PG}(3,q)$, $\mathcal{P}(\mathcal{S},x)$, where $\mathcal{S}$ is a known GQ of order $s>3$ with regular point x, and $\mathcal{P}(\mathcal{S},x)^D$ with $\mathcal{S}$ an arbitrary GQ of order s with regular point x.

We also have a classification of **I** based on the possible subconfigurations of centers of symmetry. A classification of those GQ's having both axes *and* centers of symmetry will be done separately, see below.
We do not mention each time that there are no axes of symmetry.

I.A.1. (D) *There is precisely one center of symmetry.*

(M) $\mathcal{S}(\mathcal{F})$, with $\mathcal{F}$ a (known) flock of the quadratic cone of $\mathbf{PG}(3, q)$, $\mathcal{F}$ not a linear flock, a Kantor flock, a Ganley flock or a Penttila-Williams flock.

I.A.2. (D) *There are* $k + 1$ *distinct collinear centers of symmetry,* $1 \leq k < s$.

(M) No examples known.

I.A.3. (D) *TGQ's which have one translation line.*

(M) A $T_3(\mathcal{O})^D$ with $\mathcal{O}$ a non-classical ovoid, $\mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a non-classical Ganley flock or a Penttila-Williams flock.

I.B.1. (D) *There is a point* p *which is not a center of symmetry and each line incident with* p *contains one center of symmetry. Also,* $s = t^2$.

(M) No examples known.

I.B.2. (D) *There is a point* p *which is not a center of symmetry and each line incident with* p *is incident with* t *centers of symmetry.*

(M) The Class **I.B.2** is empty.

I.B.3. (D) *There is a point* p *which is not a center of symmetry and each line incident with* p *contains* t^2 *centers of symmetry.*

(M) The Class **I.B.3** is empty.

I.C.1. (D) *There is a point* p *which is a center of symmetry and such that each line through* p *is incident with two centers of symmetry. Also,* $s = t^2$.

(M) No examples known.

I.C.2. (D) *There is a point* p *which is a center of symmetry and such that each line through* p *is incident with* $t + 1$ *centers of symmetry.*

(M) The Class **I.C.2** is empty.

I.C.3. (D) *There is a point* p *which is a center of symmetry and such that each line through* p *is incident with* $t^2 + 1 = s + 1$ *centers of symmetry.*

(M) $(\mathcal{S}(\mathcal{F}))^*$ with $\mathcal{F}$ a non-classical Kantor flock, a non-classical Ganley flock or a Penttila-Williams flock (where the translation dual is taken w.r.t. an arbitrary translation line of $\mathcal{S}(\mathcal{F})$).

I.D.1. (D) *$\mathcal{S}$ has a regular ovoid $\mathbf{O}_N$ of which each element is a center of symmetry of $\mathcal{S}$.*

(M) The Class **I.D.1** is empty.

I.D.2. (D) *Each line of the GQ is incident with precisely two centers of symmetry.*

(M) The Class **I.D.2** is empty.

I.D.3. (D) *The centers of symmetry of $\mathcal{S}$ form a dual grid $\mathcal{G}$ with parameters $t+1, t+1$.*

(M) The Class **I.D.3** is empty.

I.D.4. (D) *Each line of $\mathcal{S}$ intersects the set of centers of symmetry of $\mathcal{S}$ and each line of $\mathcal{S}$ is incident with $k+1$ centers of symmetry, $t^2 > k \geq 2$.*

(M) The Class **I.D.4** is empty.

I.D.5. (D) *The centers of symmetry of $\mathcal{S}$ form a proper subGQ $\mathcal{S}' \cong W(t)$ of order t.*

(M) The Class **I.D.5** is empty.

I.E. (D) *Every point of $\mathcal{S}$ is a center of symmetry.*

(M) $W(s)$ with s odd, $H(3, s)$.

Symmetry-Class II

Symmetry-Class II.1. (D) *There is precisely one axis of symmetry.*

(M) $\mathcal{S}(\mathcal{F})^D$, with $\mathcal{F}$ a (known) flock of the quadratic cone of $\mathbf{PG}(3, q)$, $\mathcal{F}$ not a linear flock, a Kantor flock, a Ganley flock or a Penttila-Williams flock, $T_2(\mathcal{O})^D$ with $\mathcal{O}$ not a classical oval or translation oval of $\mathbf{PG}(2, q)$.

Symmetry-Class II.2. (D) *There are $k+1$ distinct concurrent axes of symmetry, $1 \leq k < t$.*

(M) No examples known.

Symmetry-Class II.3. (D) *TGQ's which have one translation point.*

(M) A $T_2(\mathcal{O})$ with $\mathcal{O}$ a non-classical oval, a $T_3(\mathcal{O})$ with $\mathcal{O}$ a non-classical ovoid, $\mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ a non-classical Ganley flock or a Penttila-Williams flock.

Symmetry-Class III

Symmetry-Class III.1. (D) *There is a line L which is not an axis of symmetry and each point on L is incident with one axis of symmetry. Also, $t = s^2$.*

(M) No examples known.

Symmetry-Class III.2. (D) *There is a line L which is not an axis of symmetry and each point on L is incident with s axes of symmetry.*

(M) The Class **III.2** is empty.

Symmetry-Class III.3. (D) *There is a line L which is not an axis of symmetry and each point on L is incident with s^2 axes of symmetry.*

(M) The Class **III.3** is empty.

Symmetry-Class IV

Symmetry-Class IV.1. (D) *There is a line L which is an axis of symmetry and such that each point on L is incident with precisely two axes of symmetry. Also, $t = s^2$.*

(M) No examples known.

Symmetry-Class IV.2. (D) *There is a line L which is an axis of symmetry and such that each point on L is incident with $s + 1$ axes of symmetry.*

(M) The Class **IV.2** is empty.

Symmetry-Class IV.3. (D) *There is a line L which is an axis of symmetry and such that each point on L is incident with $s^2 + 1 = t + 1$ axes of symmetry.*

(M) $(\mathcal{S}(\mathcal{F})^D)^*$ with $\mathcal{F}$ a non-classical Kantor flock, a non-classical Ganley flock or a Penttila-Williams flock.

Symmetry-Class V

Symmetry-Class V.1. (D) *$\mathcal{S}$ has a regular spread $\mathbf{T}_N$ of which any line is an axis of symmetry of $\mathcal{S}$.*

(M) The Class **V.1** is empty.

Symmetry-Class V.2. (D) *Each point of the GQ is incident with precisely two axes of symmetry.*

(M) The Class **V.2** is empty.

Symmetry-Class V.3. (D) *The axes of symmetry of $\mathcal{S}$ form a grid $\mathcal{G}$ with parameters $s+1, s+1$.*

(M) The Class **V.3** is empty.

Symmetry-Class V.4. (D) *The axes of symmetry of $\mathcal{S}$ cover $\mathcal{S}$ and each point of $\mathcal{S}$ is incident with $k+1$ axes of symmetry, $s^2 > k \geq 2$.*

(M) The Class **V.4** is empty.

Symmetry-Class V.5. (D) *The axes of symmetry of $\mathcal{S}$ form a proper subGQ $\mathcal{S}' \cong \mathcal{Q}(4,s)$ of order s.*

(M) The Class **V.5** is empty.

Symmetry-Class VI

Symmetry-Class VI. (D) *Every line of $\mathcal{S}$ is an axis of symmetry.*

(M) $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$.

13.2 GQ's Having Both Centers and Axes of Symmetry

There also easily follows a classification for those GQ's having both centers and axes of symmetry. In each of the class descriptions below, we have that $s = t$.

Symmetry-Class A.1. (D) *There is a flag (p, L) so that p is a center of symmetry and L is an axis of symmetry.*

(M) No examples known.

Symmetry-Class A.2. (D) *There is an axis of symmetry L each point of which is a center of symmetry (so L is a translation line).*

(M) $T_2(\mathcal{O})^D$ with $\mathcal{O}$ not a classical oval or translation oval of $\mathbf{PG}(2,q)$.

Symmetry-Class A.3. (D) *There is a center of symmetry p each line through which is an axis of symmetry (so p is a translation point).*

(M) $T_2(\mathcal{O})$ with $\mathcal{O}$ not a classical oval or translation oval of $\mathbf{PG}(2,q)$.

Symmetry-Class A.4. (D) *There is a point p each line through which is an axis of symmetry, and there is one line $M\mathrm{I}p$ each point of which is a center of symmetry (so p is a translation point and M is a translation line).*

(M) $T_2(\mathcal{O}) \cong T_2(\mathcal{O})^D$ with $\mathcal{O}$ a non-classical translation oval of $\mathbf{PG}(2,q)$.

Symmetry-Class A.5. (D) *Each point is a center of symmetry and each line is an axis of symmetry.*

(M) $W(q)$, q even.

13.3 Classification of Span-Symmetric Generalized Quadrangles

As a corollary of the complete classification theorem, we obtain the following result on span-symmetric generalized quadrangles:

Theorem 13.3.1. *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$ and base-grid Γ. Then we have one of the following possibilities:*

(i) *we have that $s = t$ and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$, s a prime power;*

(ii) *$t = s^2$ with s a prime power, and the only axes of symmetry of $\mathcal{S}$ are contained in $\mathcal{L}$. Also, there are $s+1$ distinct subGQ's of order s, all isomorphic to $\mathcal{Q}(4,s)$ and mutually intersecting in Γ;*

(iii) *$t = s^2$ with s a prime power, and there is exactly one axis of symmetry which is not contained in $\mathcal{L}$, and this line is in $\mathcal{L}^\perp$. As in the preceding case, there are $s+1$ distinct subGQ's of order s, all isomorphic to $\mathcal{Q}(4,s)$ and mutually intersecting in Γ;*

(iv) *we are not in one of the preceding cases, and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$ with s a power of 2;*

(v) *s is odd and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, and there are at least two infinite classes of non-classical examples and one sporadic non-classical example which can occur. Also, if $\mathcal{S}$ is not classical, then $\mathcal{S}$ contains a unique line of translation points (so there is some line L so that each line of $L^\perp$ is an axis of symmetry) and s is a prime power.* ■

Chapter 14

Blueprint for the Classification of Translation Generalized Quadrangles

In the present chapter, we will describe a classification program for all translation generalized quadrangles, which is suggested "directly" by the main results of Chapter 7, Chapter 8 and Chapter 10. The blueprint thus obtained is so that some large parts of that program are completely solved in the aforementioned chapters. The blueprint was first mentioned in K. Thas [149] — see also K. Thas [151]. Section 14.2 is based on J. A. Thas and K. Thas [134].

14.1 The Classification of Translation Generalized Quadrangles, I

An essential question (and possibly the most essential question) in the classification of all translation generalized quadrangles is the determination of those TGQ's $\mathcal{S} = T(\mathcal{O})$ of order (q^n, q^m), where q is odd if $n = m$, for which $\mathcal{S} = T(\mathcal{O}) \cong \mathcal{S}^* = T(\mathcal{O}^*)$. The only known examples with that property are:

(i) For $n = m$. The $T_2(\mathcal{O})$ of Tits of order q^n, q odd (i.e. the classical GQ $\mathcal{Q}(4, q^n)$, as in that case $\mathcal{O}$ is a conic of $\mathbf{PG}(2, q^n)$);

(ii) For $n \neq m$.

(a) The $T_3(\mathcal{O})$ of Tits of order (q^n, q^{2n}).

(b) The TGQ's $\mathcal{S}(\mathcal{F})^D$, $\mathcal{F}$ a Kantor flock.

14.2 Automorphisms of TGQ's

The results of this section are all taken from J. A. Thas and K. Thas [134]. The following lemma is well known.

Lemma 14.2.1. *If G is the translation group of the TGQ $\mathcal{S}^{(x)}$, then G is a normal subgroup of the group of all automorphisms of $\mathcal{S}$ fixing x.*

Proof. Let $\phi \in G \setminus \{\mathbf{1}\}$ and let θ be an automorphism of $\mathcal{S}$ which fixes x. Then $\theta^{-1}\phi\theta$ fixes x linewise and has no fixed point in $P \setminus x^{\perp}$, with P the point set of $\mathcal{S}$. Hence $\theta^{-1}\phi\theta$ is an elation with base-point x. As G is the group of all elations with base-point x, we have $\theta^{-1}\phi\theta \in G$. ■

The following theorem is due to L. Bader, G. Lunardon and I. Pinneri (Lemma 1 of [4]), but for the sake of completeness we formulate it here for any TGQ. In [4] the result is proved with a general argument holding for coset geometries constructed from an abelian group, while here we give an alternative proof which in its turn can be applied to more general geometries.

Theorem 14.2.2. *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^m) with translation point (∞), and let $\mathbf{GF}(q)$ be a subfield of the kernel $\mathbf{GF}(q')$ of $T(\mathcal{O})$, where $\mathcal{O}$ either is a generalized ovoid ($n \neq m$) or a generalized oval ($n = m$) in $\mathbf{PG}(2n + m - 1, q) \subseteq \mathbf{PG}(2n + m, q)$. Then every automorphism of $\mathcal{S}$ which fixes (∞) is induced by an automorphism of $\mathbf{PG}(2n + m, q)$ which fixes $\mathcal{O}$, and conversely.*

Proof. Suppose $\mathcal{S}$ and $\mathcal{O}$, etc., are as above. First of all, note that any automorphism of $\mathbf{PG}(2n + m, q)$ which fixes $\mathcal{O}$ clearly induces an automorphism of $T(\mathcal{O})$ which fixes (∞). Hence, to prove the theorem, it suffices to show that any automorphism of $T(\mathcal{O})$ which fixes (∞), induces an automorphism of $\mathbf{PG}(2n + m, q)$ which fixes $\mathcal{O}$.

Consider a point x in $\mathbf{PG}(2n+m, q) \setminus \mathbf{PG}(2n+m-1, q)$, and suppose that ϕ is an automorphism of $T(\mathcal{O})$ which fixes the points (∞) and x. Denote by $\mathcal{C} = \mathcal{C}((\infty), x)$ the group of automorphisms of $T(\mathcal{O})$ which fix (∞) and x linewise. We know that $\mathcal{C}$ is isomorphic to the multiplicative group of $\mathbf{GF}(q')$, where $\mathbf{GF}(q')$ is the kernel of $T(\mathcal{O})$. Let $\mathcal{C}'$ be the subgroup of $\mathcal{C}$ induced by the perspectivities of $\mathbf{PG}(2n+m, q)$ with axis $\mathbf{PG}(2n + m - 1, q)$ and center x. Take a nontrivial $\sigma \in \mathcal{C}'$. Then $\phi^{-1}\sigma\phi$ is an element of $Aut(\mathcal{S})$ which fixes (∞) and x linewise, so belongs to $\mathcal{C}$. As $\mathcal{C}'$ is a subgroup of the cyclic group $\mathcal{C}$ and $\sigma \in \mathcal{C}'$, we have that $\sigma' = \phi^{-1}\sigma\phi \in \mathcal{C}'$.
First suppose $q \neq 2$. Consider a line L through x in $\mathbf{PG}(2n+m, q)$, and suppose y and z are points of L, both not in $\mathbf{PG}(2n+m-1, q)$, where x, y and z are distinct points, and so that $z = y^{\sigma'} = y^{\phi\sigma\phi^{-1}}$. As $z^{\phi} = (y^{\phi})^{\sigma}$ and σ fixes the affine lines of $\mathbf{AG}(2n + m, q) = \mathbf{PG}(2n + m, q) \setminus \mathbf{PG}(2n + m - 1, q)$ through x, the points x, y^{ϕ} and z^{ϕ} are on the same line of $\mathbf{AG}(2n + m, q)$, and hence ϕ maps affine lines of $\mathbf{AG}(2n + m, q)$ through x onto affine lines (through x). Now suppose θ' is an arbitrary nontrivial element of the translation group G of $T(\mathcal{O})$, and note that by

Chapter 8 of FGQ, every element of G is induced by an elation of $\mathbf{PG}(2n+m,q)$ with axis $\mathbf{PG}(2n+m-1,q)$. Then $\phi^{-1}\theta'\phi=\theta$ is also an element of the translation group; see Lemma 14.2.1. Consider an arbitrary affine line M of $\mathbf{AG}(2n+m,q)$ not through x, and suppose that $\theta'\in G$ is so that M is mapped onto some affine line of $\mathbf{AG}(2n+m,q)$ through x. Then $M^\phi=M^{\theta'\phi\theta^{-1}}$, and hence M^ϕ is also an affine line of $\mathbf{AG}(2n+m,q)$. Thus, any element ϕ of $Aut(\mathcal{S})$ which fixes (∞) and some point x of $\mathbf{PG}(2n+m,q)\setminus\mathbf{PG}(2n+m-1,q)$ induces an element of the stabilizer of $\mathbf{PG}(2n+m-1,q)$ in $\mathbf{P\Gamma L}(2n+m+1,q)$ which fixes $\mathcal{O}$. Since G is a normal subgroup of $Aut(\mathcal{S})_{(\infty)}$, we have $Aut(\mathcal{S})_{(\infty)}=GH=HG$, where H is the stabilizer in $Aut(\mathcal{S})_{(\infty)}$ of an arbitrary point in $P\setminus(\infty)^\perp$ (P is as usual). Since each element of G, respectively of H, maps affine lines of $\mathbf{AG}(2n+m,q)$ onto affine lines of $\mathbf{AG}(2n+m,q)$, the theorem follows.
Now assume $q=2$. Suppose that ϕ is an automorphism of $T(\mathcal{O})$ which fixes (∞). Let L and M be lines of $\mathbf{PG}(2n+m,2)$ not contained in $\mathbf{PG}(2n+m-1,2)$, but containing a common point $u\in\mathbf{PG}(2n+m-1,2)$. Let θ be the translation of $T(\mathcal{O})$ defined by $l_1^\theta=l_2$, with l_1,l_2 the points of L not in $\mathbf{PG}(2n+m-1,2)$. Then θ is induced by a translation of $\mathbf{AG}(2n+m,2)=\mathbf{PG}(2n+m,2)\setminus\mathbf{PG}(2n+m-1,2)$. If m_1,m_2 are the points of M not in $\mathbf{PG}(2n+m-1,2)$, then clearly $m_1^\theta=m_2$. By Lemma 14.2.1, the automorphism $\phi^{-1}\theta\phi=\theta'$ is a translation of $T(\mathcal{O})$. We have $(l_1^\phi)^{\theta'}=l_2^\phi$ and $(m_1^\phi)^{\theta'}=m_2^\phi$. Hence the lines $l_1^\phi l_2^\phi$ and $m_1^\phi m_2^\phi$ of $\mathbf{PG}(2n+m,2)$ contain a common point u' of $\mathbf{PG}(2n+m-1,2)$. If we put $u'=u^\phi$, then ϕ defines a bijection of $\mathbf{PG}(2n+m,2)$ onto itself, such that lines of $\mathbf{PG}(2n+m,2)$ not contained in $\mathbf{PG}(2n+m-1,2)$ are mapped onto lines. It easily follows that also the lines of $\mathbf{PG}(2n+m-1,2)$ are mapped onto lines. So ϕ is induced by an automorphism of $\mathbf{PG}(2n+m,2)$ which fixes $\mathcal{O}$. ∎

Theorem 14.2.3. *Suppose $\mathcal{S}_i=T(\mathcal{O}_i)$ is a TGQ of order (q^n,q^m) with translation point $(\infty)_i$ and $\mathbf{GF}(q)$ a subfield of the kernel, where $\mathcal{O}_i$ either is a generalized ovoid ($n\neq m$) or a generalized oval ($n=m$) in $\mathbf{PG}^{(i)}(2n+m-1,q)\subseteq\mathbf{PG}^{(i)}(2n+m,q)$, with $i=1,2$. Then every isomorphism of $\mathcal{S}_1$ onto $\mathcal{S}_2$ which maps $(\infty)_1$ onto $(\infty)_2$ is induced by an isomorphism of $\mathbf{PG}^{(1)}(2n+m,q)$ onto $\mathbf{PG}^{(2)}(2n+m,q)$ which maps $\mathcal{O}_1$ onto $\mathcal{O}_2$, and conversely.* ∎

The following general theorem has a proof completely similar to the proof of Theorem 14.2.2. We first need a definition. Let $\Gamma=(P,B,I)$ be a point-line incidence geometry. A *generalized linear representation* of Γ in $\mathbf{AG}(n,q)$ is defined similarly as the usual linear representation (cf. Appendix A for a formal definition of the latter) of Γ in $\mathbf{AG}(n,q)$, but where we allow subspaces of $\mathbf{AG}(n,q)$ instead of lines as in a usual linear representation. Thus, it is a monomorphism θ of Γ into the geometry of points and subspaces of the affine space $\mathbf{AG}(n,q)$, in such a way that P^θ is the set of all points of $\mathbf{AG}(n,q)$, that B^θ is a union of parallel classes of subspaces (not necessarily of the same dimension) of $\mathbf{AG}(n,q)$, and that each point of L^θ is the image of some point of L for any line L in B. We usually identify Γ with its image Γ^θ.

Theorem 14.2.4. *Let Γ be a point-line incidence geometry having a generalized linear representation θ in $\mathbf{AG}(n,q)$, $n>1$. Assume that G is a subgroup of $Aut(\Gamma)$ so that the group of translations of $\mathbf{AG}(n,q)$ is a normal subgroup of G, and so that the group of homologies of the projective completion $\mathbf{PG}(n,q)$ of $\mathbf{AG}(n,q)$ with axis $\mathbf{PG}(n-1,q)=\mathbf{PG}(n,q)\setminus\mathbf{AG}(n,q)$ and center $x\in\mathbf{AG}(n,q)$ is a normal subgroup of G_x. Then G is induced by a subgroup of $\mathbf{A\Gamma L}(n+1,q)$ fixing the set of all spaces at infinity of the elements of Γ^θ, and vice versa.* ■

In particular, Theorem 14.2.4 applies to translation planes, see Section 1 of M. Kallaher [51] (Chapter 5 of [19]).

Remark 14.2.5. Theorem 14.2.2 is also contained in C. M. O'Keefe and T. Penttila [66] for the special case of the $T_d(\mathcal{O})$ of Tits, $d\in\{2,3\}$.

We now have the following result which is an important step towards the determination of all translation generalized quadrangles.

Theorem 14.2.6. *Suppose $\mathcal{S}=T(\mathcal{O})$ is a TGQ of order (q^n,q^m) with base-point (∞), where q is odd if $n=m$, and suppose that $G_{(\infty)}$ is the stabilizer of (∞) in the automorphism group G of $\mathcal{S}$. Furthermore, suppose $(\infty)'$ is the base-point of $T(\mathcal{O}^*)=\mathcal{S}^*$, and let $G'_{(\infty)'}$ be the stabilizer of $(\infty)'$ in the automorphism group G' of $\mathcal{S}^*$. Suppose u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$, respectively. Then $[G_{(\infty)}]_u\cong[G'_{(\infty)'}]_v$. We also have that*

$$|G_{(\infty)}|=|G'_{(\infty)'}|.$$

Proof. Suppose $\mathcal{O}=\{\pi^0,\dots,\pi^{q^m}\}$ is contained in $\Pi=\mathbf{PG}(2n+m-1,q)$, and embed Π in the $(2n+m)$-space $\Pi'=\mathbf{PG}(2n+m,q)$. Let $\zeta^0,\dots,\zeta^{q^m}$ be the tangent spaces at respectively $\pi^0,\dots,\pi^m$. Now consider the dual space $(\Pi')^*$ of Π'; then the set of tangent spaces interpreted in $(\Pi')^*$, say $\mathcal{U}^*=\{\zeta^0_*,\dots,\zeta^{q^m}_*\}$, forms a set of q^m+1 spaces of dimension n which satisfy the following properties:

(1) they intersect two by two in the same fixed point z (which corresponds to Π);

(2) for distinct i,j and k, $\zeta^i_*\zeta^j_*\zeta^k_*$ has dimension $3n$;

(3) for each i, there is an $(n+m)$-space π^i_* which contains z and ζ^i_*, and so that $\pi^i_*\cap\zeta^j_*=\{z\}$ if $i\neq j$.

Now consider an arbitrary point $x\in\Pi'\setminus\Pi$. Then the corresponding hyperplane Π_x in the dual space $(\Pi')^*$ of Π' is so that $\Pi_x\cap\mathcal{U}^*$ is an egg $\mathcal{O}^*$ in Π_x, which is clearly isomorphic to the dual egg of $\mathcal{O}$ (hence the notation). Now consider an arbitrary collineation θ of $T(\mathcal{O})$ which fixes (∞) and the point x. Then by Theorem 14.2.2, θ induces a collineation of $\Pi'=\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$ and x. Now interpret θ 'naturally' as a collineation of $(\Pi')^*$ (that is, if η is an r-dimensional

space in $(\Pi')^*$ and if η^* is the corresponding $(2n+m-r-1)$-dimensional space in the dual space Π, then η^θ is the r-dimensional space of $(\Pi')^*$ which corresponds to $(\eta^*)^\theta$). Then θ fixes z, Π_x, and hence $\mathcal{O}^*$, and thus θ induces a collineation of $T(\mathcal{O}^*)$ which fixes $(\infty)'$ and z (where $(\infty)'$ is the translation point of $T(\mathcal{O}^*)$). Thus there follows easily that

$$[G_{(\infty)}]_u \cong [G'_{(\infty)'}]_v,$$

where u, respectively v, is an arbitrary point of $T(\mathcal{O})$, respectively $T(\mathcal{O}^*)$, not collinear with (∞), respectively $(\infty)'$. For any TGQ $\mathcal{S}^{(y)} = (P, B, I)$ with translation point y, translation group H and full automorphism group G, there holds, by Lemma 14.2.1, that $G_y = (G_y)_z H = H(G_y)_z$, where z is arbitrary in $P \setminus y^\perp$. As the translation groups of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$ are isomorphic, the theorem now readily follows. ■

Remark 14.2.7. Motivated by the proof of Theorem 14.2.6, it may be useful for other purposes to define the following object $\mathcal{U}$ in $\mathbf{PG}(2n+m, q)$: $\mathcal{U} = \{\zeta^0, \ldots, \zeta^{q^m}\}$ is a set of $q^m + 1$ spaces of dimension n which satisfy the following properties:

(1) the elements of $\mathcal{U}$ intersect two by two in the same fixed point x;

(2) for distinct i, j and k, $\zeta^i\zeta^j\zeta^k$ has dimension $3n$;

(3) for each i, there is an $(n+m)$-space π^i which contains ζ^i, and so that $\pi^i \cap \zeta^j = \{x\}$ if $i \neq j$.

We call $\mathcal{U}$ a *generalized ovoid cone*, respectively *generalized oval cone*, with *vertex* x if $n \neq m$, respectively $n = m$.

14.3 The Classification of Translation Generalized Quadrangles, II

From Chapter 7 and Chapter 8, we can derive that if a TGQ $\mathcal{S}$ of order (q^n, q^m) (without the restriction on q if $n = m$) has distinct translation points, then one of the following necessarily holds:

(i) $\mathcal{S} \cong \mathcal{Q}(4, q^n)$;

(ii) $\mathcal{S} \cong \mathcal{Q}(5, q^n)$;

(iii) $\mathcal{S}^* \cong \mathcal{S}(\mathcal{F})^D$ for some flock $\mathcal{F}$ (and hence $2n = m$), q odd,

and if, in the last case, $\mathcal{S}^*$ also has distinct translation points, then

(iv) $\mathcal{S} \cong \mathcal{S}^* \cong \mathcal{S}(\mathcal{F})^D$, where $\mathcal{F}$ is a Kantor flock.

In fact, the main result of Chapter 10 asserts that, conversely, each TGQ $\mathcal{S} = T(\mathcal{O})$ for which $\mathcal{S}^* \cong \mathcal{S}(\mathcal{F})^D$ for some flock $\mathcal{F}$ and where q is odd, always has distinct translation points. If $\mathcal{S}$ and $\mathcal{S}^*$ both have one translation point, where q is odd if $n = m$, then

$$Aut(\mathcal{S})_u \cong Aut(\mathcal{S}^*)_v,$$

where u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$, respectively.

Now suppose that

(TGQ1) *the TGQ's $\mathcal{S} = T(\mathcal{O})$ of order (q^n, q^m), where q is odd if $n = m$, for which $\mathcal{S} = T(\mathcal{O}) \cong \mathcal{S}^* = T(\mathcal{O}^*)$, precisely are the classical GQ $\mathcal{Q}(4, q^n)$, q odd, a $T_3(\mathcal{O})$ of Tits of order (q^n, q^{2n}), and the TGQ's $\mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ a Kantor flock;*

(TGQ2) *all TGQ's of order q^n, q even, are essentially known; that is, if $\mathcal{S}$ is a TGQ of even order q^n, then $\mathcal{S}$ arises from an oval $\mathcal{O}$ in $\mathbf{PG}(2, q^n)$ as a $T_2(\mathcal{O})$;*

(TGQ3) *if $\mathcal{S}$ and $\mathcal{S}^*$ (in the case where the latter is defined) both have one translation point, then*

$$Aut(\mathcal{S}) \cong Aut(\mathcal{S}^*);$$

(TGQ4) *if $\mathcal{S}$ and $\mathcal{S}^*$ (in the case where the latter is defined) both have one translation point, then (TGQ3) suggests that $\mathcal{S} \cong \mathcal{S}^*$.*

Then we immediately would obtain that each TGQ $\mathcal{S} = T(\mathcal{O})$ of order (q^n, q^m) is of one of the following types:

Blueprint

(C1) $\mathcal{S}$ is a $T_2(\mathcal{O})$ of Tits of order q^n, q even;

(C2) $\mathcal{S} \cong \mathcal{Q}(4, q^n)$, q odd;

(C3) $\mathcal{S}$ is a $T_3(\mathcal{O})$ of Tits of order (q^n, q^{2n}), q even;

(C4) $\mathcal{S} \cong \mathcal{Q}(5, q^n)$, q odd;

(C5) $\mathcal{S}$ is the translation dual of the point-line dual of a non-classical flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^{2n}, q^n) (so $\mathcal{O}$ is good at some element π), q odd;

(C6) $\mathcal{S}$ is the point-line dual of a non-classical flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^{2n}, q^n) (so $\mathcal{O}^*$ is good at some element π', where $\mathcal{S}^* = T(\mathcal{O}^*)$), q odd.

14.4 Corollaries

There would be several 'direct' corollaries. The high degree of difficulty of the parts of the blueprint which remain to be solved is read from the importance of those corollaries.

(1) Each TGQ has classical order, i.e., is of order s or of order (s, s^2) for some s. This is one of the most intriguing conjectures in the theory of finite (translation) generalized quadrangles, and was already verified to be true for s even. It is also an essential part of the "classical order conjecture" for finite thick generalized quadrangles, stating that for each finite GQ of order (s,t), $s \neq 1 \neq t$, s and t have a 'classical' form, namely, $t \in \{\sqrt{s}, \sqrt[3]{s^2}, s-2, s, s+2, \sqrt{s^3}, s^2\}$, where s is a prime power.

(2) There are no non-classical TGQ's of order s with s odd. Here it should certainly be noted that there are no non-classical GQ's known of order $s > 1$, s odd. To that end, it would thus be a very important observation.

(3) The only TGQ's with an odd number of points on a line arise from the $T_d(\mathcal{O})$ construction of Tits, $d = 2, 3$. This would be an impressive result, as it already is a famous open problem to classify those TGQ's $\mathcal{S} = T(\mathcal{O})$ of order (s, s^2), s even, which have a good element (see J. A. Thas and K. Thas [135] for a partial solution of this problem, and a study of TGQ's in even characteristic).

(4) Classification of generalized ovoids (and generalized ovals). If the predicted classification is true, then the only possible generalized ovoids $\mathcal{O}$ in $\mathbf{PG}(2n + m - 1, q)$ would be so that:

(a) $m = 2n$ and $\mathcal{O}$ can be interpreted as an ovoid in $\mathbf{PG}(3, q^n)$ (in the obvious sense), and in that case, at least two (and then each) element of $\mathcal{O}$ is good; also, in the even characteristic case, each generalized ovoid is obtained in this way;

(b) $\mathcal{O}$ is good at some element π;

(c) $\mathcal{O}^*$ is good at some element π'.

For generalized ovals in $\mathbf{PG}(3n - 1, q)$, there would follow that if q is odd, each such $\mathcal{O}$ arises as in (a) from a plain conic of $\mathbf{PG}(2, q^n)$; if q is even, $\mathcal{O}$ arises as in (a) from some oval in $\mathbf{PG}(2, q^n)$.

(5) Nonexistence of m-systems. Suppose $\mathcal{M}$ is a collection of totally singular subspaces of a finite classical non-degenerate polar space $\triangle$. Then $\mathcal{M}$ is called a *partial m-system* if the elements of $\mathcal{M}$ are pairwise opposite $\mathbf{PG}(m, q)$'s of $\triangle$; if a theoretical maximal bound μ_m is attained for $|\mathcal{M}|$, see [109] (this number μ_m is just the number of elements of a partition of $\triangle$ by generators), then $\mathcal{M}$ is called an *m-system*.

In *Constructions of polygons from buildings* [110], E. E. Shult and J. A. Thas (who also introduced the notion of (partial) m-systems) developed a construction method of finite geometries (with special emphasis on generalized quadrangles), using a strange subset $\mathcal{C}$ of flags of a known finite geometry and then defining new objects by a process of sequentially taking collections of new flags in various residues according to certain rules (which can be thought of as a game played on a Dynkin diagram). The 'new' geometry is denoted by $\Gamma(\mathcal{C})$. The construction of the point-line geometry $\Gamma(\mathcal{C})$ starts with a 'point' $p = F_0$, where F_0 is a flag of a spherical building $\triangle$. The stabilizer of F_0 in the automorphism group $Aut(\triangle)$ of $\triangle$ (which is a Chevalley group) is a parabolic subgroup of $Aut(\triangle)$ which induces an action on the residue $Res_{\triangle}(F_0) = \overline{\triangle}$. The kernel of this action is a semidirect product of a normal unipotent group U, possibly extended by a certain group (a scalar action centralized by the 'Levi factor', see [110]). That unipotent group U induces a group of automorphisms of $\Gamma(\mathcal{C})$ which fixes the 'point' $p = F_0$ and all 'lines' incident with it. Also, U acts regularly on the set of objects of $\Gamma(\mathcal{C})$ farthest from p.

Recall from [109] that a partial m-sytem $\mathcal{M}$ of a polar space $\triangle$ satisfies the *BLT-Property* if each line which lies on $\triangle$ intersects no more than two distinct elements of $\mathcal{M}$ nontrivially. Starting from a finite classical non-degenerate polar space $\triangle$, $\Gamma(\mathcal{C})$ is a generalized quadrangle if and only if the elements of $\mathcal{C}$ pairwise intersect at F_0, and the set $\overline{\mathcal{C}}$ corresponding to $\mathcal{C}$ in $\overline{\triangle}$ is a partial m-system with the BLT-Property. If $\Gamma(\mathcal{C})$ is a generalized quadrangle, the unipotent group U induces a group of elations with center p that acts regularly on the points not collinear with p, so that $\Gamma(\mathcal{C})$ is an EGQ with elation point p. If the group U is abelian, then it follows that $\Gamma(\mathcal{C})^{(p)}$ is a TGQ with translation point p. This is the case when $\triangle$ is a polar space defined by a quadratic form [110], and F_0 is a point of the quadric. Let us list three distinct possible cases for TGQ's which arise from the latter observation, in the notation of [110]. Below, $\mathcal{Q}^+(2n+1,q)$, $n \geq 1$, denotes a nonsingular hyperbolic quadric in $\mathbf{PG}(2n+1,q)$; $\mathcal{Q}^-(2n+1,q)$, $n \geq 2$, denotes a nonsingular elliptic quadric in $\mathbf{PG}(2n+1,q)$.

(a) $\triangle = \mathcal{Q}^+(7,q)$, $\overline{\triangle} = \mathcal{Q}^+(5,q)$, $\overline{\mathcal{C}}$ IS A 1-SYSTEM OF q^2+1 LINES IN $\mathcal{Q}^+(5,q)$, q ODD, WITH THE BLT-PROPERTY. The corresponding TGQ has order (q^2,q^2). By [109], only one such $\Gamma(\mathcal{C})$ exists; it is isomorphic to $\mathcal{Q}(4,q^2)$. Thus it is an extra motivation for the blueprint to be true in the $(s=t)$-case.

(b) $\triangle = \mathcal{Q}^+(11,q)$, $\overline{\triangle} = \mathcal{Q}^+(9,q)$, $\overline{\mathcal{C}}$ IS A 2-SYSTEM OF q^4+1 PLANES IN $\mathcal{Q}^+(9,q)$, q ODD, WITH THE BLT-PROPERTY. The corresponding TGQ would have order (q^3,q^4). No such TGQ, and whence no such 2-sytem $\overline{\mathcal{C}}$, exists if the blueprint appears to be true.

(c) $\triangle = \mathcal{Q}^-(4r-3,q)$, $\overline{\triangle} = \mathcal{Q}^-(4r-5,q)$, $\overline{\mathcal{C}}$ is an $(r-2)$-system of $q^{2r-2}+1$ $(r-2)$-dimensional spaces in $\mathcal{Q}^-(4r-5,q)$, $r \geq 3$, with the BLT-Property. The corresponding TGQ has order (q^{r-1}, q^{2r-2}). For each value of r, one such $\Gamma(\mathcal{C})$ is known; it is isomorphic to $\mathcal{Q}(5, q^{r-1})$. If it appears that two such TGQ's $\Gamma(\mathcal{C})$ and $\Gamma(\mathcal{C}')$ are isomorphic if and only if the corresponding $(r-2)$-systems $\overline{\mathcal{C}}$ and $\overline{\mathcal{C}'}$ are isomorphic (as $(r-2)$-systems of $\mathcal{Q}^-(4r-5,q)$), and if the blueprint is true, then as $\mathcal{Q}(5, q^{r-1})$ is the only GQ arising from this construction, there can be only one such $(r-2)$-system.

Appendix A

Semi Quadrangles

As already described in Chapter 6, some particular geometries arise in a natural way in the theory of symmetries of generalized quadrangles and in the theory of translation generalized quadrangles, as certain subgeometries of generalized quadrangles with concurrent axes of symmetry; these subgeometries have interesting automorphism groups. Semi quadrangles axiomatize these geometries.

In the present appendix, we will introduce 'semi quadrangles', which are finite partial linear spaces with a constant number of points on each line, having no ordinary triangles and containing, as minimal circuits, ordinary quadrangles and pentagons, with the additional property that every two non-collinear points are collinear with at least one other point of the geometry. Thick semi quadrangles generalize (thick) partial quadrangles (see [20]). We will emphasize the special situation of the semi quadrangles which are subgeometries of finite generalized quadrangles. We will present several examples of semi quadrangles, most of them arising from generalized quadrangles or partial quadrangles. We will state an inequality for semi quadrangles which generalizes the inequality of P. J. Cameron [20] for partial quadrangles, and the inequality of D. G. Higman [41, 42] for generalized quadrangles (the proof also gives information about the equality, generalizing a result of C. C. Bose and S. S. Shrikhande [14], cf. Theorem 1.4.1). Some other inequalities and divisibility conditions are computed. Also, we will characterize the linear representations of the semi quadrangles, and we will have a look at the point graphs of semi quadrangles.

This appendix is based on K. Thas [150].

A.1 Semi Quadrangles

We start with some formal definitions.
Consider an incidence structure of rank 2. As in the case of GQ's, sometimes a line is identified with the set of points incident with it, and we will do this

without further notice. The *point-line dual* or just *dual* of an incidence structure is obtained by interchanging the labels 'point' and 'line' (and by interchanging the 'corresponding' parameters). Let $\mathcal{S}$ be a point-line incidence structure. A *path of length* d is a $(d+1)$-tuple of points in which consecutive elements are distinct and collinear. Distances in an incidence structure are measured in the corresponding *incidence graph* (where adjacency is incidence). The *diameter of an incidence structure* is the diameter of its incidence graph, and a finite incidence structure is *connected* if the diameter is finite. An incidence structure is called a *partial linear space* if each point is on at least two (distinct) lines, if all lines are incident with at least two (distinct) points, and if any two distinct points are incident with at most one line (or, equivalently, if any two distinct lines are incident with at most one point). If each two distinct points of a partial linear space are collinear, then it is called a *linear space.* A *semi quadrangle (SQ)* is a partial linear space in which any line is incident with a constant number of points, which contains no ordinary triangles, but contains an ordinary quadrangle and pentagon, and every two non-collinear points are always collinear with at least one common point. It is clear that from this definition it does not necessarily follow that every point is incident with the same number of lines (such as in the case of thick generalized polygons, see [164, 1.5.3]). In order to have some more information about these structures, we introduce the *μ-parameters* and the *order* of a semi quadrangle.
Suppose that s, t_i, μ_j, where $1 \leq i \leq n$ and $1 \leq j \leq m$ for nonzero natural numbers n and m, are natural numbers satisfying $s \geq 1$ and $t_i \geq 1$. Then a semi quadrangle of *order* $(s; t_1, t_2, \ldots, t_n)$ and with *μ-parameters* $(\mu_1, \mu_2, \ldots, \mu_m)$ is an incidence structure with the following properties, where we note that (SQ1) and (SQ2) essentially define the parameters of the SQ, and that (SQ3) and (SQ4) are the main axioms.

(SQ1) The geometry is a partial linear space. Any point is incident with t_1+1,$t_2+1, \ldots$, or t_n+1 lines, and every line is incident with $s+1$ points. Also, for any $i \in \{1, 2, \ldots, n\}$ there is a point incident with t_i+1 lines.

(SQ2) If two points are not collinear, then there are exactly $\mu_1, \mu_2, \ldots$, or μ_m points collinear with both, and each of these cases occurs.

(SQ3) For any two non-collinear points there is at least one point which is collinear with both (i.e. for each $i = 1, 2, \ldots, m$ there holds that $\mu_i \geq 1$).

(SQ4) The geometry contains an ordinary pentagon and an ordinary quadrangle but no ordinary triangle as subgeometry (hence there is a j for which $\mu_j \geq 2$).

Remark A.1.1. We emphasize that (SQ2) and (SQ3) should be regarded as different axioms (instead of integrating (SQ3) in (SQ2) by demanding that for every $i = 1, 2, \ldots, m$, $\mu_i \geq 1$). For instance, suppose that $\mathcal{S}$ is a GQ of order (s, t) with $s, t > 2$, and suppose $\mathcal{L}$ is an arbitrary set of k lines with $0 < k < t$. Define a geometry by taking away the lines of $\mathcal{L}$ in the GQ, with the same points as $\mathcal{S}$ and

with the natural incidence. Then this geometry satisfies (SQ1), (SQ2) and (SQ4), but not (SQ3).
Other motivations for this distinction will be clear from the following sections, see e.g. Section A.3. Also, the reason for demanding that every line has to be incident with a constant number of points is motivated by Theorem 6.7.7, Section 6.9.1, Section 6.10 and the Examples (a) through (f), below.

As for GQ's, if a point p and a line L are incident, we simply write pIL, and if they are not incident, we write $p\not{I}L$. In the following we agree that $t_1 \leq t_2 \leq \ldots \leq t_n$ and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_m$. If there are only two possible values for the number of lines through a point, then the SQ is called *near minimal*. Since the parameters t_1, t_n, μ_1, μ_m will play an important role in the following, we call $(s; t_1, t_n)$ the *extremal order* and (μ_1, μ_m) the *extremal μ-parameters*. For a near minimal semi quadrangle, the order and the extremal order coincide. A semi quadrangle is called *thick* if every point is incident with at least three lines and if every line is incident with at least three points. A thick semi quadrangle with $t_1 = t_2 = \ldots = t_n = t$ and $\mu_1 = \mu_2 = \ldots = \mu_m = \mu$ is a thick *partial quadrangle (PQ)* — as defined by P. J. Cameron in [20] — with *parameters* (s, t, μ) and with $\mu \neq 1$ (the latter notation differs somewhat from that of P. J. Cameron, but in this context it is more convenient), and a thick partial quadrangle with $\mu = t + 1$ is precisely a thick generalized quadrangle with parameters (s, t). Thick generalized quadrangles always contain quadrangles and pentagons. In the case of generalized quadrangles, the condition that the GQ must contain a pentagon is equivalent with the thickness of the GQ, see [164, §1.3]. This is not the case for semi quadrangles; there are geometries with only two points per line which satisfy all the SQ-conditions. For example, define the geometry $\Gamma = (P, B, I)$ as follows. The point set P consists of six distinct 'letters' a_i, $i \in \{1, 2, \ldots, 6\}$, lines are the sets $\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_5, a_6\}, \{a_6, a_2\}, \{a_1, a_5\}, \{a_5, a_4\}$, and incidence is the natural one. Then $\mathcal{S}$ is a semi quadrangle. Also, every *strongly regular graph with parameters* (v, k, λ, μ) (see e.g. Chapter 22 of [19]) and with $\mu \geq 2$ and $\lambda = 0$ is a semi quadrangle of order $(1; k-1)$ and with μ-parameters (μ) (and is hence also a partial quadrangle). An example is the unique strongly regular graph with parameters $(16, 5, 0, 2)$, namely the *Clebsch graph*, see [26, p. 440].

Remark A.1.2. A thick semi quadrangle $\mathcal{S}$ is a thick generalized quadrangle if and only if the following property is satisfied:

(GQ3) *Consider a point p and a line L, p not incident with L. Then there is exactly one line which intersects L and which is incident with p.*

Proof. Immediate. ■

Notation. Suppose that A is a set of points, respectively lines, of an SQ $\mathcal{S}$. Then, following GQ notation, $A^\perp$ is the set of points, respectively lines, of $\mathcal{S}$ which are collinear, respectively concurrent, with every point, respectively line, of A.

Other standard GQ notation will sometimes be taken over without causing confusion.

The following theorem shows that a semi quadrangle is loaded with pentagons.

Theorem A.1.3. *Any anti-flag (a non-incident point-line pair) (p, L) of a semi quadrangle $\mathcal{S}$ which does not satisfy Property (GQ3) is contained in a pentagon.*

Proof. Suppose (p, L) is an anti-flag which does not satisfy Property (GQ3). Suppose u is a point of L and that $x \in \{u, p\}^{\perp}$. Since p is not collinear with any point on L, there is a point v on L that is not collinear with any point on px. Let $y \in \{v, p\}^{\perp}$. Then $y \notin xp, xu$ and hence $uvypx$ is a pentagon which contains L and p. ■

Theorem A.1.4. *A geometry $\mathcal{G}$ which satisfies all the SQ-conditions except that there must be a pentagon, automatically contains pentagons if and only if it is not a grid or a dual grid.*

Proof. It is clear that grids which are GQ's and dual grids satisfy all the SQ-conditions except the conditions that there must be a pentagon. If the geometry $\mathcal{G}$ is not a grid or a dual grid and if it satisfies (GQ3), then by Remark A.1.2, the geometry is a thick generalized quadrangle and hence there are always pentagons. If the geometry $\mathcal{G}$ does not satisfy (GQ3), then it cannot be a grid or a dual grid and applying Theorem A.1.3, the proof is complete. ■

A.2 A Motivation for Introducing Semi Quadrangles

In alignment of Section 6.10 of Chapter 6, we now obtain

Theorem A.2.1. *Suppose that $\mathcal{S}' = (P', B', I')$ is a subgeometry of a GQ $\mathcal{S} = (P, B, I)$ of order (s, t), with the properties that there are $s' + 1$ points on each line for some s', that there is an ordinary subpentagon in $\mathcal{S}'$ and that (SQ3) is satisfied. Then $s \neq 1 \neq t$, and two points of $\mathcal{S}'$ are collinear if and only if they are collinear in $\mathcal{S}$. If $s' = s$, then $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order (s, t') with $t' \neq 1$.*

Proof. That $s \neq 1 \neq t$ follows from the fact that $\mathcal{S}'$ has a pentagon. Suppose p and q are collinear points of $\mathcal{S}'$. Then p and q are also collinear points in $\mathcal{S}$ trivially. Next, suppose p and q are points of $\mathcal{S}'$ which are collinear in $\mathcal{S}$ but not in $\mathcal{S}'$. Then by (SQ3) there is a point x in $\mathcal{S}'$ which is collinear with both p and q. This implies that pxq is a triangle in $\mathcal{S}$, a contradiction; hence pq is a line in $\mathcal{S}'$. Whence two points of $\mathcal{S}'$ are collinear if and only if they are collinear in $\mathcal{S}$. If $s = s'$, then by Theorem 1.3.2 it follows that $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order (s, t') with $t' \neq 1$. ■

A.3 Examples and Constructions of Semi Quadrangles

We only give examples of thick semi quadrangles which are not (always) partial quadrangles, and which are near minimal. All of them are in some way related to generalized quadrangles or partial quadrangles.
We first of all emphasize again that it should be noted that (SQ3) is a very important condition. This will be clearly reflected in the following examples.

(a) Suppose that $\mathcal{S} = (P, B, I)$ is a generalized quadrangle of order (s, t) with $s, t \geq 3$, and suppose p is a point of $\mathcal{S}$ with the property that for every two non-collinear points q, q' of $P \setminus p^{\perp}$ there holds that

(M) $|\{p, q, q'\}^{\perp}| < t + 1$.

By 1.7.1 of FGQ there is a pair of non-collinear points (x, y) in $P \setminus p^{\perp}$ with

(Q) $|\{p, x, y\}^{\perp}| < t$.

Now define the following incidence structure $\mathcal{S}_p = (P_p, B_p, I_p)$: (a) P_p is the set of points of $P \setminus p^{\perp}$, (b) B_p is the set of all lines of $\mathcal{S}$ not incident with p, and (c) I_p is the restriction of I to $(P_p \times B_p) \cup (B_p \times P_p)$. Then $\mathcal{S}_p$ is a thick semi quadrangle with s points on every line and $t + 1$ lines through every point; Condition (M) implies that (SQ3) is satisfied, (Q) implies the existence of quadrangles and Theorem A.1.4 yields the existence of a pentagon.

If $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t) with $s, t > 2$ and p an antiregular point, then the geometry $\mathcal{S}_p$ always satisfies Condition (M) and hence Condition (Q), thus $\mathcal{S}_p$ is a semi quadrangle, of which the μ-parameters are contained in $\{t - 1, t, t + 1\}$.

Now specialize, and suppose that $\mathcal{S}^{(x)}$ is a translation generalized quadrangle of order (s, t) with $s, t > 2$ and with translation point x. If $s = t$, we furthermore suppose that s is odd. Then Conditions (M) and (Q) are satisfied, see Chapter 8 of FGQ, and hence $\mathcal{S}^{(x)}$ yields a thick semi quadrangle with a constant number of lines through a point. The semi quadrangles which arise from translation generalized quadrangles in the way described above all have the property that there is an elementary abelian automorphism group (the latter defined in the usual way) which acts regularly on the points of the semi quadrangle. Also, s and t are powers of the same prime p, and there is an odd natural number a and an integer n for which $t = p^{n(a+1)}$ and $s = p^{na}$ if $s \neq t$. If $s = t$ with s odd, then by Theorem 1.4.4(v) the μ-parameters are given by $(s - 1, s + 1)$; if $s \neq t$ and if p and a are as above, then the (possible) μ-parameters are $(p^{n(a+1)} - p^n, p^{n(a+1)})$, and $\mathcal{S}_x$ is a partial quadrangle if and only if $a = 1$, and then $\mu = p^{2n} - p^n$.

Let $\mathcal{S}$ be a GQ of order (s, s^2) with $s > 2$. Then by Theorem 1.4.1 every triad of points has $s + 1$ centers (see Section A.4 for a similar result on SQ's). Now take an arbitrary point p of $\mathcal{S}$, and consider the geometry $\mathcal{S}_p$. Then $\mathcal{S}_p$ is a partial quadrangle with parameters $(s - 1, s^2, s^2 - s)$.

(b) Let $\mathcal{S}$ be a GQ of order (s,t) with $s,t > 2$, and suppose that $\mathcal{S}'$ is a subGQ of order $(s, t/s)$, with the property that for every two non-collinear points x and y of $\mathcal{S} \setminus \mathcal{S}'$, $|\{x,y\}^\perp \cap \mathcal{S}'| < t+1$. Every line of $\mathcal{S}$ intersects $\mathcal{S}'$ in 1 or $s+1$ points ($\mathcal{S}'$ is a *geometrical hyperplane* of $\mathcal{S}$). Next, define a geometry $\mathcal{S}_{\mathcal{S}'} = (P_{\mathcal{S}'}, B_{\mathcal{S}'}, I_{\mathcal{S}'})$ where $B_{\mathcal{S}'}$ is the set of lines of $\mathcal{S}$ which are not contained in $\mathcal{S}'$, $P_{\mathcal{S}'}$ is the set of points of $\mathcal{S} \setminus \mathcal{S}'$, and where $I_{\mathcal{S}'}$ is the natural incidence. Then $\mathcal{S}_{\mathcal{S}'}$ is a thick semi quadrangle of order $(s-1;t)$.

If we put $\mathcal{S} = W(s)$ with $s > 2$ even, and $\mathcal{S}'$ is an $(s+1) \times (s+1)$-grid in $\mathcal{S}$, then $\mathcal{S}_{\mathcal{S}'}$ is an example of this construction.

(c) Suppose $\mathcal{S}$ is a GQ of order (s,t) with $s,t > 2$, and suppose that $\mathcal{O}$ is an ovoid with the property that for every two non-collinear points x and y of $\mathcal{S} \setminus \mathcal{O}$ there holds that $|\{x,y\}^\perp \cap \mathcal{O}| < t+1$. Define a geometry $\mathcal{S}_{\mathcal{O}} = (P_{\mathcal{O}}, B_{\mathcal{O}}, I_{\mathcal{O}})$ where $B_{\mathcal{O}}$ is the line set of $\mathcal{S}$, $P_{\mathcal{O}} = \mathcal{S} \setminus \mathcal{O}$, and where $I_{\mathcal{O}}$ is the natural incidence. Then $\mathcal{S}_{\mathcal{O}}$ is a thick semi quadrangle of order $(s-1;t)$.

Suppose $\mathcal{O}$ is an ovoid of the classical GQ $W(q)$ of order q, $q > 2$. Then every point of $\mathcal{S}$ is regular. By Theorem 1.9.2, $\mathcal{S}_{\mathcal{O}}$ is a semi quadrangle of order $(q-1;q)$ and with μ-parameters $(q-1, q+1)$.

(d) Suppose $\Gamma = (P,B,I)$ is a partial quadrangle with parameters (s,t,μ), where $s,t \geq 3$, and let $\Gamma' = (P', B', I')$ be a *sub partial quadrangle (subPQ)* (in the obvious sense) of Γ with parameters (s,t',μ'). Then a simple counting argument shows that every line of Γ intersects Γ' if and only if $|P'| \times (t-t') = |B| - |B'|$, that is, if and only if

$$\begin{aligned}&(t-t')(s+1)(1+(t'+1)s(1+\frac{st'}{\mu'}))\\&=(1+(t+1)s(1+\frac{st}{\mu}))(t+1)-(1+(t'+1)s(1+\frac{st'}{\mu'}))(t'+1).\end{aligned} \tag{A.1}$$

Note that if we interchange the words ‘PQ’ and ‘GQ’, then this condition can be simplified to $t' = t/s$, see Example (b).
Assume Condition (A.1) is satisfied. Furthermore, we suppose that Γ has the property that (1) for every two non-collinear points q, q' of $P \setminus P'$ there holds that $|\{q,q'\}^\perp \cap \Gamma'| < \mu$, and that (2) there is a pair of non-collinear points (x,y) in $P \setminus P'$ for which $|\{x,y\}^\perp \cap \Gamma'| < \mu - 1$. Define a geometry $\Gamma_{\Gamma'} = (P_{\Gamma'}, B_{\Gamma'}, I_{\Gamma'})$ where $B_{\Gamma'}$ is the line set of $\Gamma \setminus \Gamma'$ and $P_{\Gamma'}$ is the set of points of $\Gamma \setminus \Gamma'$, and where $I_{\Gamma'}$ is the natural incidence. Then $\Gamma_{\Gamma'}$ is a semi quadrangle of order $(s-1;t)$.

(e) A *partial ovoid* of a partial quadrangle is a set of mutually non-collinear points. An *ovoid* $\mathcal{O}$ of a partial quadrangle Γ with parameters (s,t,μ) is a set of non-collinear points such that every line is incident with exactly one point of the set. Dually, one defines *partial spreads* and *spreads*[1].

[1]In the same way, one could define *(partial) ovoids* for semi quadrangles, and, dually, *(partial) spreads* .

By counting the point-line pairs (p, L) of Γ for which $p \in \mathcal{O}$ and pIL with L a line of Γ, in two ways, we obtain $|\mathcal{O}| = \frac{s^2t(t+1)/\mu+(t+1)s+1}{s+1}$. Suppose Γ is a (proper) PQ with parameters (s, t, μ) with $s, t > 2$, and suppose that $\mathcal{O}$ is an ovoid with the property that for every two non-collinear points x and y of $\Gamma \setminus \mathcal{O}$ there holds that $|\{x, y\}^\perp \cap \mathcal{O}| < \mu$. Also, we demand that there is a pair of non-collinear points (x, y) in $\Gamma \setminus \mathcal{O}$ for which $|\{x, y\}^\perp \cap \mathcal{O}| < \mu - 1$. Define a geometry $\Gamma_\mathcal{O} = (P_\mathcal{O}, B_\mathcal{O}, I_\mathcal{O})$ where $B_\mathcal{O}$ is the line set of Γ, $P_\mathcal{O} = \Gamma \setminus \mathcal{O}$, and where $I_\mathcal{O}$ is the natural incidence. Then $\Gamma_\mathcal{O}$ is a thick semi quadrangle of order $(s-1; t)$. We do not know of any concrete examples of such semi quadrangles; we only prove the following nonexistence theorem for ovoids of partial quadrangles.

Theorem A.3.1. *Consider a partial quadrangle Γ with parameters $(s-1, s^2, s^2-s)$, $s > 2$. Then Γ cannot have ovoids.*

Proof. Suppose that $\mathcal{O}$ is an ovoid of Γ. Then

$$|\mathcal{O}| = \frac{(s-1)^2s^2(s^2+1)/(s^2-s) + (s^2+1)(s-1) + 1}{s} = s^3.$$

By A. A. Ivanov and S. V. Shpectorov [48], Γ is of the form $\mathcal{S}_p$ (see Example (a)), where $\mathcal{S}$ is a GQ of order (s, s^2), and p is a point of $\mathcal{S}$. Hence $\mathcal{O} \cup \{p\}$ is an ovoid of $\mathcal{S}$, a contradiction by Theorem 1.9.1. ■

Note. By the appendix of Chapter 4 of [149] it easily follows that if $\mathcal{O}$ is a partial ovoid of the PQ $\mathcal{S}_p$, then $|\mathcal{O}| \leq s^3 - s^2 + s$.

(f) Suppose $\mathcal{K}$ is a *complete* $(t+1)$*-cap* of $\mathbf{PG}(n-1, q)$, $n \geq 3$ (see Section A.6), and embed $\mathbf{PG}(n-1, q)$ in $\mathbf{PG}(n, q)$. Suppose P is the set of points of $\mathbf{PG}(n, q)$ which are not contained in $\mathbf{PG}(n-1, q)$, that B is the set of lines L of $\mathbf{PG}(n, q)$ which are not contained in $\mathbf{PG}(n-1, q)$ and for which $|\mathcal{K} \cap L| = 1$. Then the geometry $\mathcal{S} = (P, B, I)$, with I the natural incidence, is a semi quadrangle of order $(q-1; t)$. If $n = 4$ and $\mathcal{K}$ is an ovoid of $\mathbf{PG}(3, q)$, then $\mathcal{S}$ is a partial quadrangle, cf. [20]. For details and proofs, see Section A.6.

Remark A.3.2. (i) The first three constructions given above all arise by taking away a geometrical hyperplane of a GQ. Thus these geometries are all AGQ's. Also, any of the Examples (a),(b),(c),(d),(e) can clearly be generalized in a natural way by considering *geometrical hyperplanes of semi quadrangles* (instead of *geometrical hyperplanes of partial quadrangles*).

(ii) The Examples (a), (b) and (c) are the only thick semi quadrangles which are subgeometries of a GQ with the same number of points on a line as the GQ, minus one.

A.4 Computation of Some Divisibility Conditions, Constants and Inequalities

If a point p of a semi quadrangle is incident with $t_i + 1$ lines, then we denote this by $p \in P_i$, and p is said to have *degree* t_i.
If we write $\lfloor x \rfloor$, with $x \in \mathbb{R}$, then we mean the greatest natural number which is at most x, and with $\lceil x \rceil$ we mean the smallest natural number which is at least x.

A.4.1 Some generalities

Suppose $\mathcal{S} = (P, B, I)$ is a thick semi quadrangle of order $(s; t_1, t_2, \ldots, t_n)$ and with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$ which is not a generalized quadrangle. Then by the observation we made earlier, $\mathcal{S}$ does not satisfy (GQ3), and hence there is a point-line pair (p, L) so that $p \not{I} L$ and for which there is no line M satisfying $pIM \sim L$. Suppose $p \in P_j$. By counting the points which are collinear with p and with a point of L in two ways, we obtain $(s+1)\mu_1 \leq (t_j + 1)s$, from which follows that $\mu_1 + \frac{\mu_1}{s} \leq t_j + 1$.
Now suppose that $t_i = t$ for all i, and fix a point q. By $N_k = N_k(q)$, we denote the number of points x of $P \setminus q^\perp$ for which there are μ_k points collinear with both x and q. Now we count the number of points in $q^\perp \setminus \{q\}$ in two ways, and we get that

$$\frac{N_1\mu_1 + N_2\mu_2 + \ldots + N_m\mu_m}{st} = (t+1)s, \tag{A.2}$$

hence we have the following theorem.

Theorem A.4.1. *Suppose that $\mathcal{S}$ is a semi quadrangle with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$ and with a constant number, $t+1$, of lines through a point. Suppose the N_i are as above. Then*

$$\frac{N_1\mu_1 + N_2\mu_2 + \ldots + N_m\mu_m}{st} = (t+1)s. \tag{A.3}$$

■

A.4.2 An inequality for semi quadrangles

Definition. Suppose $\mathcal{S}$ is a semi quadrangle. Following GQ theory, a *triad* is a set of three points, respectively lines, two by two non-collinear, respectively non-concurrent. A *center* of a triad $\{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$, where $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ are all points or all lines, is an element of $\{\mathcal{U}, \mathcal{V}, \mathcal{W}\}^\perp$.

Theorem A.4.2. *Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m). Then we have the following inequality:*

$$\begin{aligned}[(t_1 - 1)s\mu_1]^2 \leq \mu_m[(\mu_m - 1)(\mu_m - 2) + (t_n - 1)s](\lfloor \frac{(t_n + 1)t_n s^2}{\mu_1} \rfloor \\ -s(t_1 + 1) + \mu_m - 1). \end{aligned} \tag{A.4}$$

If equality holds, then there is a constant $x_0 = \frac{(t_1-1)s\mu_1}{\lfloor \frac{(t_n+1)t_n s^2}{\mu_1} \rfloor - s(t_1+1)+\mu_m-1}$ *such that each triad of points has exactly* x_0 *centers.*
Also, if each triad of points has a constant number of centers, then

$$[(t_n-1)s\mu_m]^2 \geq \mu_1[(t_1-1)s+(\mu_1-1)(\mu_1-2)](\lceil \frac{(t_1+1)t_1s^2}{\mu_m} \rceil - s(t_n+1)+\mu_1-1). \tag{A.5}$$

Proof. Suppose p is a point of degree t_1 of $\mathcal{S}$. We repeat our general assumption that $t_1 \leq t_n$ and $\mu_1 \leq \mu_m$. There are $(t_1+1)s$ points collinear with, and different from, p and if p' is such a point, then there are at least $t_1 s$ points collinear with, and different from, p' and not collinear with p. Since $\mu_1 \leq \mu_m$, there are at least $\frac{(t_1+1)t_1s^2}{\mu_m}$ points not collinear with p.
If q is a point not collinear with p, then there are at most $s(t_n+1)-\mu_1$ points collinear with q and not with p, but different from q. So, there are at least $a = \lceil \frac{(t_1+1)t_1s^2}{\mu_m} \rceil - s(t_n+1)+\mu_1-1$ points not collinear with p and q. Analogously there are at most $a'' = \lfloor \frac{(t_n+1)t_ns^2}{\mu_1} \rfloor - s(t_1+1)+\mu_m-1$ points not collinear with both p and q.
We suppose a' is the *precise* number of points not collinear with p and q. Suppose $p_1, p_2, \ldots, p_{a'}$ are these a' points, and suppose x_i is the number of points collinear with p, q and p_i $(1 \leq i \leq a')$. Then we have the following inequalities:

$$b = (t_1-1)s\mu_1 \leq \sum_{i=1}^{i=a'} x_i \leq (t_n-1)s\mu_m = b' \tag{A.6}$$

(since for each of the at most μ_m points collinear with p and q there are at most $(t_n-1)s$ choices for p_i, and for each of the at least μ_1 points collinear with p and q there are at least $(t_1-1)s$ choices for p_i);

$$\mu_1(\mu_1-1)(\mu_1-2) \leq \sum_{i=1}^{i=a'} x_i(x_i-1) \leq \mu_m(\mu_m-1)(\mu_m-2) \tag{A.7}$$

(since for each pair of distinct points collinear with p and q there are at most μ_m-2 choices for p_i and at least μ_1-2 choices for p_i).
It follows that

$$c = \mu_1(\mu_1-1)(\mu_1-2)+(t_1-1)s\mu_1 \leq \sum_{i=1}^{i=a'} x_i^2 \tag{A.8}$$

and

$$\sum_{i=1}^{i=a'} x_i^2 \leq \mu_m(\mu_m-1)(\mu_m-2)+(t_n-1)s\mu_m = c'. \tag{A.9}$$

So, first of all,

$$\sum_{i=1}^{i=a'}(x_i - x)^2 = \sum_{i=1}^{i=a'} x^2 - 2\sum_{i=1}^{i=a'} xx_i + \sum_{i=1}^{i=a'} x_i^2 \leq a'x^2 - 2bx + c'. \tag{A.10}$$

Since the left-hand side is a positive semi-definite quadratic form, we can conclude that

$$b^2 \leq a'c' \leq a''c', \tag{A.11}$$

and $b^2 \leq a''c'$ gives the first inequality of the theorem.
If equality holds, then with $a' = a''$ and $x_0 = \frac{b}{a'} = \frac{c'}{b} = \frac{b}{a''}$, we have that

$$\sum_{i=1}^{i=a'}(x_i - x_0)^2 = 0,$$

and so $x_i = x_0$ for $i = 1, 2, \ldots, a'$.
Now we clearly also have the following inequality:

$$ax^2 - 2b'x + c \leq \sum_{i=1}^{i=a'}(x_i - x)^2. \tag{A.12}$$

Since (A.5) is trivial if $a \leq 0$, we suppose that $a > 0$. If we suppose that $x_i = x_0$ for a certain constant x_0 and every i, then we have that

$$ax_0^2 - 2b'x_0 + c \leq 0, \tag{A.13}$$

and since $a > 0$, we now know that this quadratic form has at least one real root. Hence ${b'}^2 > ac$, which yields the complete proof of the theorem. ■

Note. If equality holds in (A.4), we have the divisibility conditions $a''|b$ and $b|c'$, with $a'' = a'$, b and c' defined as above.

Remark A.4.3. (i) If the dual of the SQ $\mathcal{S}$ is also a semi quadrangle, then the dual statement of Theorem A.4.2 also holds. In that case $t_1 = t_2 = \ldots = t_n = t$, and for every two non-concurrent lines there is at least one line concurrent with both.

(ii) In P. J. Cameron [20] it is proved that a partial quadrangle of order (s, t, μ) has the property that the dual is also a partial quadrangle if and only if $t = s$ or $t + 1 = \mu$.

(iii) One notes that we used (SQ3) implicitly in the proof of Theorem A.4.2, since we divided by μ_1. Also, we did not 'completely' use (SQ4), hence Theorem A.4.2 also holds for several other incidence geometries.

Corollary A.4.4 (P. J. Cameron [20]). *Suppose $\mathcal{S}$ is a partial quadrangle with parameters (s, t, μ). Then*

$$\mu(t-1)^2 s^2 \leq [s(t-1) + (\mu-1)(\mu-2)][\frac{(t+1)ts^2}{\mu} - (t+1)s + \mu - 1]. \quad \text{(A.14)}$$

Equality holds if and only if the number of points collinear with each of any three pairwise non-collinear points is a constant; if this occurs, the constant is $1 + \frac{(\mu-1)(\mu-2)}{s(t-1)}$. ■

Corollary A.4.5 (D. G. Higman [41, 42]). *Suppose $\mathcal{S}$ is a generalized quadrangle with parameters (s, t), $s \neq 1 \neq t$. Then $t \leq s^2$ and, dually, $s \leq t^2$.* ■

Corollary A.4.6 (C. C. Bose and S. S. Shrikhande [14]). *Let $\mathcal{S}$ be a generalized quadrangle with parameters (s, t), $s \neq 1 \neq t$. Then $t = s^2$ if and only if the number of points collinear with each of any three pairwise non-collinear points is a constant, and if this occurs, the constant is $s + 1$. Dually, $s = t^2$ if and only if the number of lines concurrent with each of any three pairwise non-concurrent lines is a constant, and if this occurs, the constant is $t + 1$.* ■

A.4.3 Some inequalities

Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m), and assume that $s \leq t_1$. Suppose b is the number of lines and v is the number of points. Counting the number θ of *flags* of $\mathcal{S}$ (a *flag* is an incident point-line pair), we get that

$$v(t_n + 1) \geq \theta = b(s+1) \geq v(t_1 + 1). \quad \text{(A.15)}$$

Note. If $t_1 = t_n = t$ in (A.15), then $v(t+1) = b(s+1)$. If $v = b$, then $t_n \geq s \geq t_1$.

In the case $v = b$, some refinement is possible.

Theorem A.4.7. *Suppose $\mathcal{S} = (P, B, I)$ is a semi quadrangle of order $(s; t_1, t_2, \ldots, t_n)$ and with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$. If $\mathcal{S}$ has the property that $v := |P| = |B| =: b$, then we have that either $t_1 = t_n = s$ or $t_1 < s < t_n$.*

Proof. Suppose $\mathcal{S}$ is an SQ with equally many points as lines, and suppose $t_1 \neq t_n$ (so $t_1 < t_n$). If we suppose that M_i is the number of points with degree t_i, and if we count the number θ of flags of $\mathcal{S}$, we obtain the following.

$$\theta = b(s+1) = v(s+1) = \sum_i M_i(t_i + 1).$$

Since $\sum_i M_i = v$, the theorem easily follows. ■

Theorem A.4.8. *Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m). Then we have the following inequalities:*

$$s^2 t_1(t_1+1) \leq (v-(t_1+1)s-1)\mu_m, \tag{A.16}$$

and

$$s^2 t_n(t_n+1) \geq (v-(t_n+1)s-1)\mu_1, \tag{A.17}$$

and $\mathcal{S}$ is a PQ if and only if equality holds in both (A.16) and (A.17).

Proof. The inequalities are immediate by counting in two ways the ordered triples of points (p, q, r) of $\mathcal{S}$, with the property that p, q and r are not on the same line, and that $p \sim q$ and $p \sim r$. If $\mathcal{S}$ is a PQ, then $t_1 = t_n = t$, $\mu_1 = \mu_m = \mu$ and $s^2 t(t+1) = (v-(t+1)s-1)\mu$. If equality holds in (A.16) and (A.17), then from $s^2 t_1(t_1+1) \leq s^2 t_n(t_n+1)$ and $(v-(t_1+1)s-1)\mu_m \geq (v-(t_n+1)s-1)\mu_1$ follows that $s^2 t_1(t_1+1) = s^2 t_n(t_n+1) = (v-(t_n+1)s-1)\mu_1 = (v-(t_1+1)s-1)\mu_m$, and so $t_1 = t_n$ and $\mu_1 = \mu_m$, that is, $\mathcal{S}$ is a partial quadrangle. ■

A.5 Semi Quadrangles and their Point Graphs

A (simple) *graph* is an incidence structure in which lines are called *edges* and points are called *vertices*, and in which any edge is incident with two points and any two distinct points are incident with at most one edge. Two distinct points incident with the same edge are called *adjacent*, and a graph is *complete* if any two distinct vertices are adjacent. If a vertex v is incident with t edges, then t is called the *valency* of v. The *μ-values* of a graph $\mathcal{G}$ are numbers $\mu_1, \mu_2, \ldots, \mu_m$ such that any two non-adjacent vertices are both adjacent with μ_i vertices for some $1 \leq i \leq m$. The *λ-values* of a graph are numbers $\lambda_1, \lambda_2, \ldots, \lambda_{m'}$ such that any two distinct adjacent points are both adjacent with λ_j points for some $1 \leq j \leq m'$. An *induced subgraph* consists of a subset of points of the point set, together with all the edges joining two points in the subset, and a *(maximal) clique* is a (maximal) complete induced subgraph of a graph. The *point graph* of an incidence geometry is the graph in which two distinct points are adjacent if and only if they are collinear (where the vertices are the points of the geometry).

Theorem A.5.1 (P. J. Cameron [20]). *The point graph of a partial quadrangle is strongly regular and has no induced subgraph isomorphic to a complete graph on four points with one edge removed. Conversely, a strongly regular graph with this property is the point graph of a partial quadrangle.* ■

There is a similar theorem for semi quadrangles.

Theorem A.5.2. *A graph is the point graph of a semi quadrangle if and only if* (a) *every μ-value is strictly positive (i.e. the diameter of the graph is at most* 2*),* (b) *there is only one λ-value $s-1$, and* (c) *the graph contains an induced quadrangle, respectively pentagon, and it has no induced subgraph isomorphic to a*

complete graph on four points with one edge removed. Moreover, if the SQ has order $(s; t_1, t_2, \ldots, t_n)$ *and* μ*-parameters* $(\mu_1, \mu_2, \ldots, \mu_m)$*, then the* λ*-value of the graph is* $s-1$*, the possible* μ*-values are* $\mu_1, \mu_2, \ldots, \mu_m$*, and* $\{(t_1+1)s, (t_2+1)s, \ldots, (t_n+1)s\}$ *is the set of valencies. Conversely, a graph which satisfies Properties* (a), (b) *and* (c)*, and which has these parameters, is the point graph of a semi quadrangle of order* $(s; t_1, t_2, \ldots, t_n)$ *and with* μ*-parameters* $(\mu_1, \mu_2, \ldots, \mu_m)$*.*

Proof. Suppose $\mathcal{S}$ is a semi quadrangle of order $(s; t_1, t_2, \ldots, t_n)$ and with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$. It immediately follows that its point graph has valencies $(t_1+1)s, (t_2+1)s, \ldots, (t_n+1)s$, that the λ-value is $s-1$, and that $\{\mu_1, \mu_2, \ldots, \mu_m\}$ is the set of μ-values. If there would be an induced subgraph isomorphic to a complete graph on four points p, q, p', q' with one edge $p'q'$ removed, then p' and q' (as points of $\mathcal{S}$) both lie on the line pq, and so they are collinear, a contradiction. The other conditions of (SQ4) are reflected in (c). Now suppose a graph $\mathcal{G}$ has one λ-value $s-1$ and μ-values $\mu_1, \mu_2, \ldots, \mu_m$, and suppose that it satisfies Properties (a) and (c). Any edge pq is contained in a unique maximal clique $\mathcal{G}_{pq}$ whose vertex set consists of p, q and all vertices joined to both. Hence $\mathcal{G}_{pq}$ has $s+1$ vertices. If any vertex is contained in t_i+1 maximal cliques, with $i \in \{1, 2, \ldots, n\}$, then it is clear that the vertices and maximal cliques of $\mathcal{G}$ form a semi quadrangle of order $(s; t_1, t_2, \ldots, t_n)$ and with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$. ■

A.6 Linear Representations

A *linear representation* of a semi quadrangle $\mathcal{S} = (P, B, I)$ is a monomorphism θ of $\mathcal{S}$ into the geometry of points and lines of the affine space $\mathbf{AG}(n, q)$, in such a way that P^θ is the set of all points of $\mathbf{AG}(n, q)$, that B^θ is a union of parallel classes of lines of $\mathbf{AG}(n, q)$, and that each point of L^θ is the image of some point of L for any line L in B. Usually we identify $\mathcal{S}$ with its image $\mathcal{S}^\theta$. Note that any parallel class of lines partitions the point set of $\mathbf{AG}(n, q)$. Since parallel classes of lines in an $\mathbf{AG}(n, q)$ correspond to points of $\mathbf{PG}(n-1, q)$ in a natural way, such a representation $\mathcal{S}^\theta$ defines a set of points $\mathcal{K}$ in $\mathbf{PG}(n-1, q)$. An *r-cap* in $\mathbf{PG}(n-1, q)$ (usually called *r-arc* if $n = 3$) is a set of r points, no three of which are collinear. A line is *secant*, respectively *tangent*, to (or of) an r-cap according as it meets the cap in two points, respectively one point.

Theorem A.6.1 (P. J. Cameron [20]). (1) *A subset* $\mathcal{K}$ *of the point set of* $\mathbf{PG}(n-1, q)$ *provides a linear representation of a partial quadrangle with parameters* $(q-1, t, \mu)$ *if and only if it is a* $(t+1)$*-cap with the property that any point not in* $\mathcal{K}$ *lies on* $t-\mu+1$ *tangents to* $\mathcal{K}$*.*

(2) *A subset* $\mathcal{K}$ *of the point set of* $\mathbf{PG}(n-1, q)$ *provides a linear representation of a generalized quadrangle* $\mathcal{S}$ *if and only if one of the following occurs:*

(a) $n = 2$ *and* $|\mathcal{K}| = 2$*;*

(b) $n = 3$, *q is even and $\mathcal{K}$ is a hyperoval;*

(c) $q = 2$ *and $\mathcal{K}$ is the complement of a hyperplane.* ■

Remark A.6.2. If $n = 2$ and $|\mathcal{K}| = 2$, then $\mathcal{S}$ is a grid. If $q = 2$ and $\mathcal{K}$ is the complement of a hyperplane, then $\mathcal{S}$ is a dual grid. If $n = 3$, $q > 2$, and $\mathcal{K}$ is a hyperoval, then $\mathcal{S}$ is neither a grid nor a dual grid.

Now suppose $\mathcal{S} = (P, B, I)$ is a semi quadrangle with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_k)$, and suppose $\mathcal{S}$ has a linear representation in an $\mathbf{AG}(n, q)$. If $t + 1$ is the number of parallel classes defined by this representation, then it is first of all clear that $\mathcal{S}$ is of order $(q - 1; t)$ (hence every point of $\mathcal{S}$ is incident with a constant number of lines).
Suppose $\mathcal{V}$ is the set of $t + 1$ points of $\mathbf{PG}(n - 1, q)$ which corresponds to the semi quadrangle, and suppose that three points p, o and r of $\mathcal{V}$ are collinear. Consider an arbitrary affine point x of $\mathbf{AG}(n, q)$, and suppose $y \neq x$ is a point of $\mathbf{AG}(n, q) \cap xr$. Then the lines yp, xo and xr define a triangle which is contained in the semi quadrangle, a contradiction. Hence $\mathcal{V}$ is a $(t + 1)$-cap.

Let $\mathcal{K}$ be the $(t+1)$-cap in $\mathbf{PG}(n-1, q)$ which corresponds to a semi quadrangle $\mathcal{S}$, and suppose p and o are arbitrary points of $\mathbf{AG}(n, q)$ which are non-collinear in $\mathcal{S}$. Then the line po of $\mathbf{PG}(n, q)$ intersects $\mathbf{PG}(n - 1, q)$ in a point r off $\mathcal{K}$. Suppose there are μ_j points of $\mathcal{S}$ collinear (in $\mathcal{S}$) with p and o. Then this means that there are exactly $\mu_j/2$ planes through pq in the projective completion $\mathbf{PG}(n, q)$ of $\mathbf{AG}(n, q)$ which intersect $\mathcal{K}$ in exactly two points, and hence there are precisely $t - \mu_j + 1$ tangents to $\mathcal{K}$ through r.
Since $|\mathbf{AG}(n, q)| = |P|$, *every* point of $\mathbf{PG}(n - 1, q)$ off $\mathcal{K}$ is incident with μ_h tangent lines to $\mathcal{K}$ for a certain $h \in \{1, 2, \ldots, k\}$. It follows that $\mu_h \equiv 0 \mod 2$ for all feasible h. Now suppose o' is such a point which is incident with $t - \mu + 1$ tangents to $\mathcal{K}$ ($\mu \in \{\mu_1, \mu_2, \ldots, \mu_k\}$), and let L be an arbitrary line through o' and not in $\mathbf{PG}(n-1, q)$. Then every two distinct points x, y on L, $x \neq o' \neq y$, are non-collinear in $\mathcal{S}$, and $|\{x, y\}^\perp| = \mu$. Also, there are exactly $\frac{q(q-1)}{2}$ such (non-ordered) pairs on L.

Condition (SQ3). The fact that $\mathcal{S}$ satisfies (SQ3) is clearly equivalent with the fact that for every two points p and o of $\mathbf{AG}(n, q)$ for which po does not intersect $\mathcal{K}$, there must be at least one secant to the cap through $po \cap \mathbf{AG}(n, q)$. Hence the cap must be *complete* (by definition).

We now investigate how the existence of a quadrangle, respectively pentagon, is reflected on the linear representation.

The existence of a quadrangle. Let L be a secant to $\mathcal{K}$ and let π be a plane of $\mathbf{PG}(n, q)$ containing L, but not contained in $\mathbf{PG}(n - 1, q)$. Then π contains quadrangles of $\mathcal{S}$.

The existence of a pentagon. By Theorem A.1.3 and the preceding results, the existence of a pentagon in $\mathcal{S}$ is equivalent to the condition that $\mathcal{S}$ is not a

grid or a dual grid. By Remark A.6.2, this is always the case except if one of the following occurs:

(1) $n = 2$ and $|\mathcal{K}| = 2$;

(2) $q = 2$ and $\mathcal{K}$ is the complement of a hyperplane.

We have proved the following theorem.

Theorem A.6.3. (1) *A subset $\mathcal{K}$ of the point set of $\mathbf{PG}(n-1, q)$, $n \geq 3$, provides a linear representation of a semi quadrangle with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_k)$ if and only if the following conditions are satisfied:*

(a) *it is a complete $(t+1)$-cap for a certain t with the property that any point off $\mathcal{K}$ in $\mathbf{PG}(n-1, q)$ lies on $t - \mu_j + 1$ tangents to $\mathcal{K}$ for some $\mu_j \in \{\mu_1, \mu_2, \ldots, \mu_k\}$, and each possibility occurs;*

(b) *if $q = 2$, then $\mathcal{K}$ is not the complement of a hyperplane.*

(2) *If a $(t+1)$-cap $\mathcal{K}$ of $\mathbf{PG}(n-1, q)$ provides a linear representation of the semi quadrangle $\mathcal{S}$, then every point of $\mathcal{S}$ is incident with $t + 1$ lines.*

(3) *Suppose $\mathcal{S} = (P, B, I)$ is an SQ with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_k)$ which has a linear representation in $\mathbf{AG}(n, q)$, and define P_j by $P_j = \{\{x, y\}$ with $(x, y) \in P \times P$, $x \not\sim y \parallel |\{x, y\}^{\perp}| = \mu_j\}$. Then for all j, $|P_j| \equiv 0$ mod $(q(q-1)/2)$. Also, each μ_h is even $\forall h$.* ■

If $n = 3$ and q is even, then the $(t+1)$-cap $\mathcal{K}$ is a hyperoval if $t+1$ equals $q+2$, and a hyperoval is always complete. Suppose $q \geq 4$. If we consider the semi quadrangle $\mathcal{S}$ which corresponds with $\mathcal{K}$, then $\mathcal{S}$ is the $T_2^*(\mathcal{K})$ due to R. W. Ahrens and G. Szekeres (see also Theorem A.6.1). If $n = 4$, $q > 2$, and $\mathcal{K}$ is an ovoid (which is a complete (q^2+1)-cap) of $\mathbf{PG}(3, q)$, then the associated semi quadrangle is a partial quadrangle with parameters $(q-1, q^2, q^2-q)$.

A.7 Semi Quadrangles and Complete Caps of Projective Spaces

The following two theorems are direct consequences of Theorem A.6.3 and the results of Section A.4.

Theorem A.7.1. *Suppose $\mathcal{K}$ is a complete $(k+1)$-cap in $\mathbf{PG}(n, q)$, where $n \geq 2$ and where $\mathcal{K}$ is not the complement of a hyperplane if $q = 2$, and let μ, respectively μ', be the integers such that every point of $\mathbf{PG}(n, q) \setminus \mathcal{K}$ is incident with at least $k+1-\mu$, respectively at most $k+1-\mu'$, tangents to $\mathcal{K}$. Then we have the following inequality:*

$$[(k-1)(q-1)\mu']^2 \leq \mu[(\mu-1)(\mu-2) + (k-1)(q-1)](\lfloor \frac{(k+1)k(q-1)^2}{\mu'} \rfloor$$
$$-(q-1)(k+1) + \mu - 1).$$

If equality holds, then there is a constant $x_0 = \frac{(k-1)(q-1)\mu'}{\lfloor\frac{(k+1)k(q-1)^2}{\mu'}\rfloor-(q-1)(k+1)+\mu-1}$ *such that, if we embed* $\mathbf{PG}(n,q)$ *as a hyperplane in* $\mathbf{PG}(n+1,q)$*, then for every set* $\{p_1,p_2,p_3\}$ *of three distinct points in* $\mathbf{PG}(n+1,q)\setminus\mathbf{PG}(n,q)$ *with the property that* $p_ip_j\cap\mathcal{K}=\emptyset$ *for every* $i\neq j$ *in* $\{1,2,3\}$*, there holds that there are precisely* x_0 *points* r *in* $\mathbf{PG}(n+1,q)\setminus\mathbf{PG}(n,q)$ *for which* $|rp_i\cap\mathcal{K}|=1$ $\forall i=1,2,3$*.*
Also, if for every set $\{p_1,p_2,p_3\}$ *of three distinct points in* $\mathbf{PG}(n+1,q)\setminus\mathbf{PG}(n,q)$ *with the property that* $p_ip_j\cap\mathcal{K}=\emptyset$ *for every* $i\neq j$ *in* $\{1,2,3\}$*, it is true that there is a constant number of points* r *in* $\mathbf{PG}(n+1,q)\setminus\mathbf{PG}(n,q)$ *for which* $|rp_i\cap\mathcal{K}|=1$ $\forall i=1,2,3$*, then we have the following:*

$$[(k-1)(q-1)\mu]^2 \geq \mu'[(k-1)(q-1)+(\mu'-1)(\mu'-2)](\lceil\frac{(k+1)k(q-1)^2}{\mu}\rceil$$
$$-(q-1)(k+1)+\mu'-1).$$

■

We do not know whether Theorem A.7.1 yields new information about complete caps in projective spaces.

Theorem A.7.2. *Suppose that* $\mathcal{K}$ *is a complete* $(k+1)$*-cap in* $\mathbf{PG}(n,q)$*, where* $n\geq 2$ *and where* $\mathcal{K}$ *is not the complement of a hyperplane if* $q=2$*, and let* μ*, respectively* μ'*, be the integers such that every point of* $\mathbf{PG}(n,q)\setminus\mathcal{K}$ *is incident with at least* $k+1-\mu$*, respectively at most* $k+1-\mu'$*, tangents to* $\mathcal{K}$*. Then we have the following inequalities:*

$$(q-1)^2k(k+1) \leq (q^{n+1}-(k+1)(q-1)-1)\mu, \tag{A.18}$$

and

$$(q-1)^2k(k+1) \geq (q^{n+1}-(k+1)(q-1)-1)\mu', \tag{A.19}$$

and equality holds in both cases if and only if $\mu=\mu'$*.* ■

It follows that

$$k+1-\frac{(q-1)^2k(k+1)}{(q^{n+1}-(k+1)(q-1)-1)} \leq k+1-\mu'. \tag{A.20}$$

From [45], we know that since $\mathcal{K}$ is complete, there holds that $k+1-\mu' < \delta(q^{n-1}+q^{n-2}+\ldots+1-k)$, where $\delta=1$ if q is even and $\delta=2$ otherwise. We conclude that, with $f(k,q,n) := \frac{(q-1)^2k(k+1)}{(q^{n+1}-(k+1)(q-1)-1)}-1$,

$$(1+\delta)k < f(k,n,q)+\delta(q^{n-1}+q^{n-2}+\ldots+1). \tag{A.21}$$

For an excellent survey on bounds of complete caps in projective spaces and several related problems, see J. W. P. Hirschfeld and L. Storme [44].

Addendum: Semi $2N$-Gons

A semi quadrangle contains no substructure isomorphic to an ordinary 2-gon or 3-gon. With this property in mind, we could define a *semi $2N$-gon* of *order* $(s; t_1, t_2, \ldots, t_n)$ and with *μ-parameters* $(\mu_1, \mu_2, \ldots, \mu_m)$ to be an incidence structure satisfying (SQ1), and also the following conditions.

(1) There is no substructure isomorphic to an ordinary M-gon, for $2 \leq M \leq 2N-1$.

(2) If two distinct points are not contained in a path of length $N-1$ or less, then they are contained in exactly $\mu_1, \mu_2, \ldots$, or μ_m paths of length N, where $\mu_j \geq 1$ for every j. Also, each of the cases occurs.

(3) There are substructures isomorphic to an ordinary $2N$-gon and an ordinary $(2N+1)$-gon.

With this definition, a (thick) *partial $2N$-gon* [20] is just a semi $2N$-gon with $s > 1$, $t_1 = t_2 = \ldots = t_n > 1$ and $\mu_1 = \mu_2 = \ldots = \mu_m$. Also, a generalized $2N$-gon is precisely a semi $2N$-gon with $t_i = \mu_j$ for arbitrary i and j.

Appendix B

The Sizes of Some Groups and a Table of Schur Multipliers

This short addendum provides the reader with tables of the sizes of the groups which were used most throughout this work, and of the Schur multipliers (defined below) of the groups with a finite split BN-pair of rank 1 that are generated by their root groups.

B.1 A Table of the Sizes of Some Groups

In Table B.1 are listed the sizes of some groups of the form $X(r, q')$, respectively $G(q')$, $q' = p^h$ with p prime, which frequently arise in this work; X denotes one of $\mathbf{P\Gamma L}$, $\mathbf{PGL}$, $\mathbf{PSL}$, $\mathbf{PSU}$, $\mathbf{SL}$, $\mathbf{SU}$; G denotes one of $\mathbf{R}$, $\mathbf{Sz}$. If $X \in \{\mathbf{PSU}, \mathbf{SU}\}$, then $q' = q^2$; in all the other cases $q' = q$.

B.2 A Table of Schur Multipliers

If G is a perfect group and $(\overline{G}, \eta)$ is its universal central extension, then $ker(\eta)$ is sometimes also called the *Schur multiplier* of G. In Table B.2, we list the Schur multiplier of a group of the form $X(r, q)$, respectively $G(q)$, $q = p^h$, as $R \times P$, where R is a p'-group and P is a p-group with $p \neq p'$, see D. Gorenstein [35] (see also J. L. Alperin and D. Gorenstein [2], R. L. Griess, Jr. [36] and I. Schur [104]). Here, the groups $X(r, q)$ and $G(q)$ are the Lie groups of which the (size of the) Schur multiplier was explicitly used in the body of this work (so $X \in \{\mathbf{PSL}, \mathbf{PSU}\}$; $G \in \{\mathbf{R}, \mathbf{Sz}\}$).

B.1. The Sizes of Some Groups.

Group	Order
$\mathbf{P\Gamma L}(r,q)$	$h(q-1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-1)$
$\mathbf{PGL}(r,q)$	$(q-1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-1)$
$\mathbf{PSL}(r,q)$	$(gcd(r,q-1))^{-1}(q-1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-1)$
$\mathbf{PSU}(r,q^2)$	$(gcd(r,q+1))^{-1}(q+1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-(-1)^i)$
$\mathbf{SL}(r,q)$	$(q-1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-1)$
$\mathbf{SU}(r,q^2)$	$(q+1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-(-1)^i)$
$\mathbf{Sz}(q)\cong {}^2\mathbf{B}_2(q)$	$(q^2+1)q^2(q-1)$
$\mathbf{R}(q)\cong {}^2\mathbf{G}_2(q)$	$(q^3+1)q^3(q-1)$

B.2. Table of Schur Multipliers.

Group	R (if $P = \{\mathbf{1}\}$)	$R \times P$ (if $P \neq \{\mathbf{1}\}$)	$(r-1, q)$
$\mathbf{PSL}(r,q)$	$\mathbb{Z}_{gcd(r,q-1)}$	$\mathbb{Z}_2$ $\mathbb{Z}_2 \times \mathbb{Z}_3$ $\mathbb{Z}_2$ $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ $\mathbb{Z}_2$	$(1,4)$ $(1,9)$ $(2,2)$ $(2,4)$ $(3,2)$
$\mathbf{PSU}(r,q^2)$	$\mathbb{Z}_{gcd(r,q+1)}$	$\mathbb{Z}_2$ $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$(3,2)$ $(3,3)$ $(5,2)$
$\mathbf{Sz}(q) \cong {}^2\mathbf{B}_2(q)$	$\{\mathbf{1}\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2,8)$
$\mathbf{R}(q) \cong {}^2\mathbf{G}_2(q)$	$\{\mathbf{1}\}$		

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Explanatory Notation Index

We list the notations and (mathematical) abbreviations which are frequently and/or systematically used (abbreviations that are used in the explanation of a notation can also be found in this index).

$\mathbf{A\Gamma L}(n+1,q)$	The full (semilinear) automorphism group of $\mathbf{AG}(n,q)$
$\mathbf{AG}(n,q)$	The n-dimensional affine space over $\mathbf{GF}(q)$
A_n	The alternating group of degree n
$Aut(\mathcal{S})$	The full automorphism group of the GQ $\mathcal{S}$
$cl(p,q)$	$\{z \in \mathcal{S} \parallel z^\perp \cap \{p,q\}^{\perp\perp} \neq \emptyset\}$
EGQ	Elation generalized quadrangle
$\mathcal{F}$	A flock of the quadratic cone in $\mathbf{PG}(3,q)$
$\mathbb{F}$	A (finite) field
$gcd(n,m)$	The greatest common divisor of the natural numbers n and m
$\mathbf{GF}(q)$	The finite (Galois) field with q elements
$G \times H$	The direct product of the groups G and H
$\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$	The base-grid of an SPGQ
G'	The derived group of G
GQ	Generalized quadrangle
$H(4,q^2)$	The classical Hermitian quadrangle of order (q^2,q^3)
$H(4,q^2)^D$	The point-line dual of the classical Hermitian quadrangle of order (q^2,q^3)
H_q^n	The dual net with $q+1$ points on any line and q^{n-1} lines through any point which satisfies the Axiom of Veblen
HPMGQ	Half pseudo Moufang generalized quadrangle

$H(3,q^2)$	The classical Hermitian quadrangle of order (q^2,q)
I	Incidence relation
$(\mathcal{J},\mathcal{J}^*)=\mathcal{J}$	A 4-gonal family
$\mathbb{K}$	A (general finite) field, or the kernel of some TGQ
$\mathcal{K}$	The quadratic cone of $\mathbf{PG}(3,q)$, or a complete $(st-t/s)$-arc in a GQ of order (s,t)
$\mathcal{L}$	The base-span of an SPGQ
$\mathbb{N}$	The set of natural numbers (including 0)
$N_H(H')$	The normalizer of H' in H (where $H'\leq H$)
$\mathcal{N}_L$	The net which arises from the regular line L of a GQ
$\mathcal{N}_p$	The net which arises from the regular point p of a GQ
$\mathcal{N}_L^*$	The point-line dual of $\mathcal{N}_L$
$\mathcal{N}_p^*$	The point-line dual of $\mathcal{N}_p$
$\mathbb{N}_0$	The set of strictly positive natural numbers
$\mathbf{O}$	An ovoid
$\mathcal{O}=\mathcal{O}(n,m,q)$	A generalized ovoid, respectively generalized oval, in $\mathbf{PG}(2n+m-1,q)$, where $n\neq m$, respectively $n=m$
$\mathcal{O}^*=\mathcal{O}^*(n,m,q)$	The dual of $\mathcal{O}=\mathcal{O}(n,m,q)$
$\mathcal{O}_x$	The ovoid subtended by x
$\mathbf{PGL}(n+1,q)$	The full linear automorphism group of $\mathbf{PG}(n,q)$
$\mathbf{P\Gamma L}(n+1,q)$	The full (semilinear) automorphism group of $\mathbf{PG}(n,q)$
$\mathbf{PG}(n,q)$	The n-dimensional projective space over $\mathbf{GF}(q)$
$[p,L]$	The unique line through p which is concurrent with L ($p \not{\mathrm{I}} L$)
π_L, Π_L	The projective plane which arises from the regular line L in a GQ of order s
π_p, Π_p	The projective plane which arises from the regular point p in a GQ of order s
$proj_L p$	The unique point on L which is collinear with p ($p \not{\mathrm{I}} L$)
$proj_p L$	The unique line through p which is concurrent with L ($p \not{\mathrm{I}} L$)
PQ	Partial quadrangle
$\mathbf{PSL}(n+1,q)$	The projective special linear subgroup of $\mathbf{PGL}(n+1,q)$
$\mathbf{PSU}(n+1,q)$	The projective special unitary subgroup of $\mathbf{PGL}(n+1,q)$
$\mathcal{P}(\mathcal{S},x)$	The GQ which arises from the GQ $\mathcal{S}$ of order s with regular point x
$\mathcal{Q}(5,q)$	The classical GQ of order (q,q^2) arising from a nonsingular elliptic quadric in $\mathbf{PG}(5,q)$

$\mathcal{Q}(4,q)$	The classical GQ of order (q,q) arising from a nonsingular parabolic quadric in $\mathbf{PG}(4,q)$
$\mathcal{Q}(3,q)$	The classical GQ of order $(q,1)$ arising from a nonsingular hyperbolic quadric in $\mathbf{PG}(3,q)$
$\mathbf{R}(q)$	The Ree group of degree q^3+1 in characteristic 3
$\mathcal{S}^D$	The point-line dual of $\mathcal{S}$
SEGQ	Strong elation generalized quadrangle
$\mathcal{S}(\mathcal{F})$	The flock GQ which corresponds to the flock $\mathcal{F}$
$\mathcal{S}(G_*)$	The closure of G_*
$\mathbf{SL}(n+1,q)$	The special linear subgroup of $\mathbf{GL}(n+1,q)$
S_n	The symmetric group of degree n
$\mathcal{S}=(P,B,I)$	A GQ with point set P, line set B, and incidence relation I
SPGQ	Span-symmetric generalized quadrangle
SQ	Semi quadrangle
$\mathcal{S}^*$	The translation dual of $\mathcal{S}$
$\mathcal{S}_\theta$	The subGQ fixed elementwise by θ
STGQ	Skew translation generalized quadrangle
$\mathbf{SU}(n+1,q)$	The special unitary subgroup of $\mathbf{GL}(n+1,q)$
$\mathcal{S}^{(x)}=(\mathcal{S}^{(x)},G)$	An EGQ or TGQ with base-point x (and base-group G)
$\mathbf{Sz}(q)$	The Suzuki group of degree q^2+1
$\mathbf{T}$	A spread
TGQ	Translation generalized quadrangle
$\mathbf{T}(\mathcal{L},M)$	The spread determined by the base-span $\mathcal{L}$ and the line M
$T(n,m,q)$	A TGQ which can be represented in $\mathbf{PG}(2n+m-1,q)$
$T(\mathcal{O})$	The TGQ which arises from the generalized ovoid (respectively oval) $\mathcal{O}$
tr	Trace function
$T^*(\mathcal{O})=T(\mathcal{O}^*)$	The translation dual of $T(\mathcal{O})$
$T_3(\mathcal{O})$	The GQ of Tits which arises from the ovoid $\mathcal{O}$ in $\mathbf{PG}(3,q)$
$T_2(\mathcal{O})$	The GQ of Tits which arises from the oval $\mathcal{O}$ in $\mathbf{PG}(2,q)$
$V^\perp$	The set of elements which are collinear, respectively concurrent, with each element of V, where $V\subseteq P$, respectively $V\subseteq B$
$V^{\perp\perp}$	The set of elements which are collinear, respectively concurrent, with each element of $V^\perp$, where $V\subseteq P$, respectively $V\subseteq B$
$W(q)$	The classical symplectic quadrangle of order q
(X,G)	A permutation group, where G acts on X
$Z(G)$	The center of the group G

Miscellaneous Notations

$\sim$	Collinear, or concurrent (or adjacent)
$\cong$	Isomorphic
$\lfloor x \rfloor$	The greatest natural number which is at most x
$\lceil x \rceil$	The smallest natural number which is at least x